ECE 2205: Electromagnetic Fields & Waves - Lecture 17

Plane Electromagnetic Waves

Reflection of a Plane Wave at Oblique Incidence

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Reflection of a Plane Wave at Oblique Incidence

Based on the general preliminaries on oblique incidence, two special cases will be considered:

A. Parallel (P) Polarization: **E** field parallel to the plane of incidence.

B. Perpendicular (S) Polarization: **E** field perpendicular to the plane of incidence.

Parallel Polarization

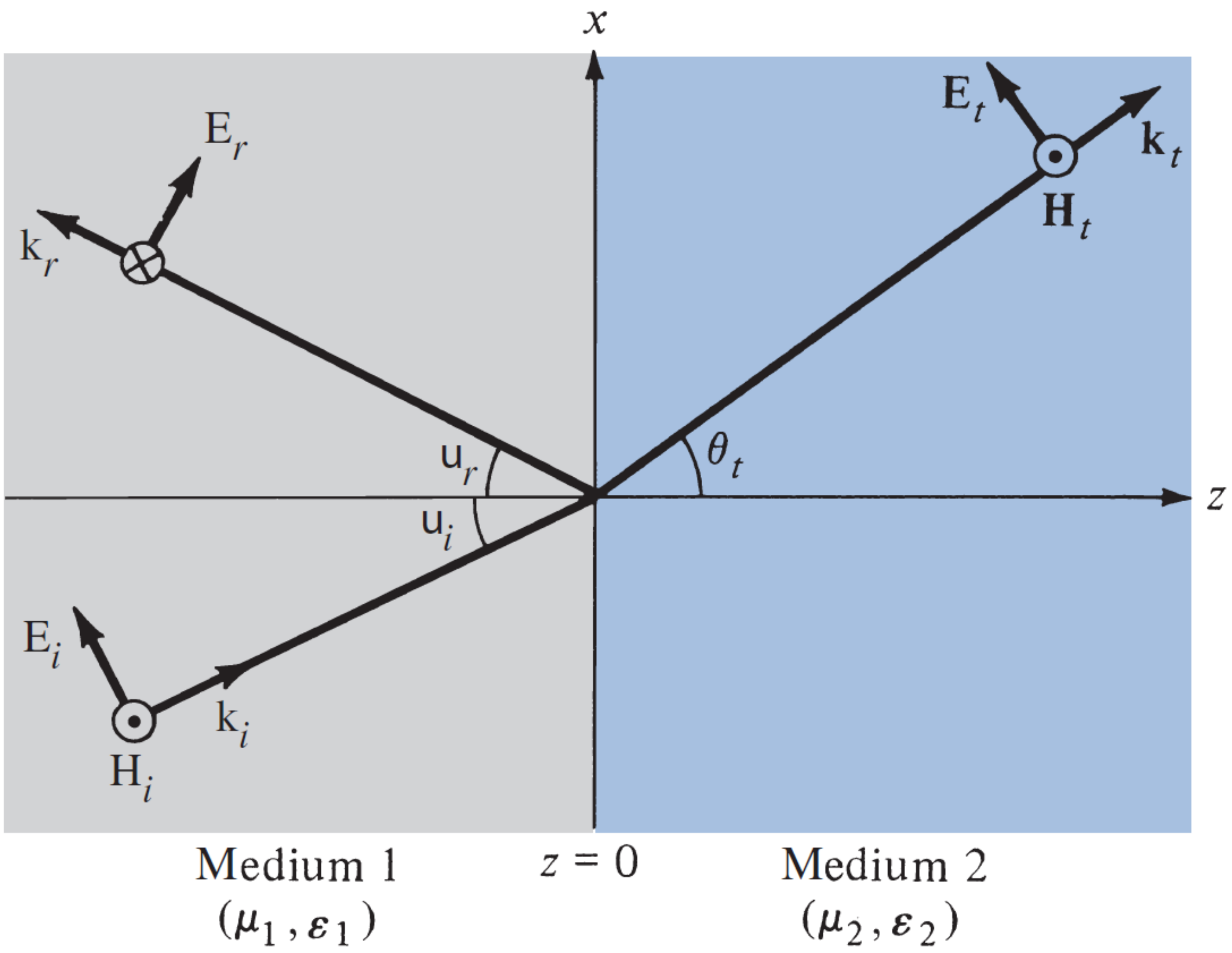


Figure 1: Oblique incidence with **E** parallel to the plane of incidence.

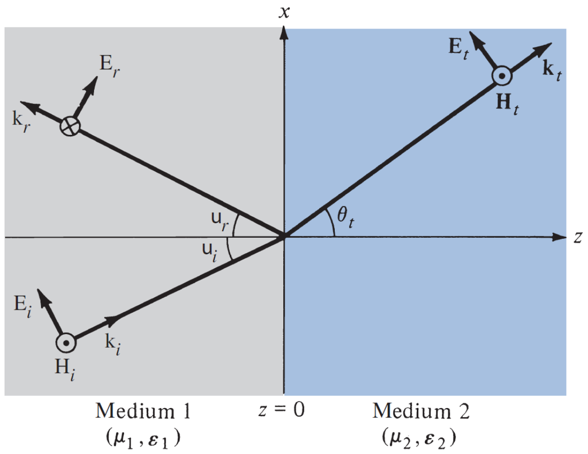
Figure 1, where the **E** field lies in the *xz*-plane, the plane of incidence, illustrates the case of parallel polarization. In medium 1, we have both incident and reflected fields given by

$$\mathbf{E}_{is} = E_{io}(\cos\theta_i\,\mathbf{a}_x - \sin\theta_i\,\mathbf{a}_z)\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \quad (1a)$$

$$\mathbf{H}_{is} = \frac{E_{io}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\mathbf{a}_y \quad (1b)$$

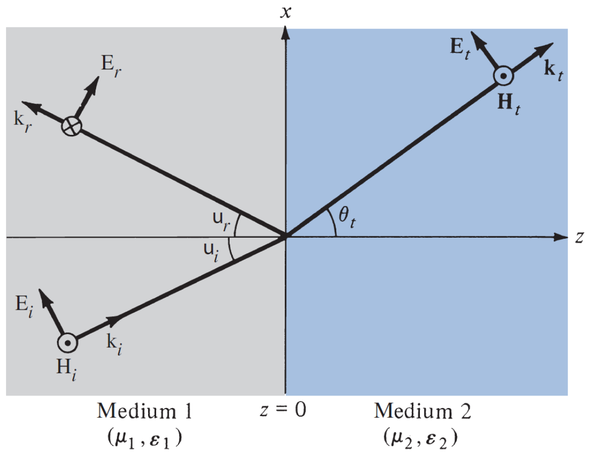
$$\mathbf{E}_{rs} = E_{ro}(\cos\theta_r\,\mathbf{a}_x + \sin\theta_r\,\mathbf{a}_z)\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \quad (1c)$$

$$\mathbf{H}_{rs} = -\frac{E_{ro}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\mathbf{a}_y \quad (1d)$$



where $\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1}$. Notice carefully how we arrive at each field component.

Once $\mathbf{k}$ is known, we define $\mathbf{E}_s$ such that $\nabla \cdot \mathbf{E}_S = 0$ or $\mathbf{k} \cdot \mathbf{E}_S = 0$ and then $\mathbf{H}_S$ is obtained from $\mathbf{H}_S = \frac{\mathbf{k}}{\omega\mu} \times \mathbf{E}_S = \mathbf{a}_k \times \frac{\boldsymbol{E}}{\eta}$.



The transmitted fields exist in medium 2 and are given by

$$\mathbf{E}_{ts} = E_{to}(\cos\theta_t\,\mathbf{a}_x - \sin\theta_t\,\mathbf{a}_z)\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \tag{2a}$$

$$\mathbf{H}_{ts} = \frac{E_{to}}{\eta_2}\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\mathbf{a}_y \tag{2b}$$

where $\beta_2 = \omega\sqrt{\mu_2 \varepsilon_2}$

Requiring that $\theta_r = \theta_i$ and that the tangential components of **E** and **H** be continuous at the boundary $z = 0$, we obtain

$$(E_{io} + E_{ro}) \cos\theta_i = E_{to} \cos\theta_t \quad (3a)$$

$$\frac{1}{\eta_1}(E_{io} - E_{ro}) = \frac{1}{\eta_2} E_{to} \quad (3b)$$

Expressing E_{ro} and E_{to} in terms of E_{io}, we obtain

$$\boxed{\Gamma_{\parallel} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}} \quad (4a)$$

or

$$\boxed{E_{ro} = \Gamma_{\parallel} E_{io}} \quad (4b)$$

$$\tau_{\|} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \quad (5a)$$

or

$$E_{to} = \tau_{\|} E_{io} \quad (5b)$$

Equations (4) and (5) are called Fresnel's equations. $\Gamma_{\|}$ and $\tau_{\|}$ are known as Fresnel coefficients. Note that the equations reduce to the equations obtained for normal incidence when $\theta_i = \theta_t = 0$ as expected. Since θ_i and θ_t are related according to Snell's law of the form:

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_i}{k_t} = \frac{u_2}{u_1} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}$$

eqs. (4) and (5) can be written in terms of θ_i by substituting

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - (u_2/u_1)^2\sin^2\theta_i} \tag{6}$$

From eqs. (4) and (5), it is easily shown that

$$\boxed{1 + \Gamma_{\|} = \tau_{\|}\left(\frac{\cos\theta_t}{\cos\theta_i}\right)} \tag{7}$$

From eq. (4a), it is evident that it is possible that $\Gamma_{\|} = 0$ because the numerator is the difference of two terms. Under this condition, there is no reflection ($E_{ro} = 0$), and the incident angle at which this takes place is called the ***Brewster angle*** $\theta_{B_{\|}}$. The Brewster angle is also known as the polarizing angle because an arbitrarily polarized incident wave will be reflected with only the component of E perpendicular to the plane of incidence.

The Brewster angle is obtained by setting $\theta_i = \theta_{B_\parallel}$ when $\Gamma_\parallel = 0$ in eqs. (4) that is,

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B_\parallel}$$

or

$$\eta_2^2(1 - \sin^2\theta_t) = \eta_1^2(1 - \sin^2\theta_{B_\parallel})$$

$$\boxed{\sin^2\theta_{B_\parallel} = \frac{1 - \mu_2\varepsilon_1/\mu_1\varepsilon_2}{1 - (\varepsilon_1/\varepsilon_2)^2}} \qquad (8)$$

It is of practical value to consider the case when the dielectric media are not only lossless but nonmagnetic as well—that is, $\mu_1 = \mu_2 = \mu_0$. For this situation, eq. (8) becomes

$$\sin^2\theta_{B_\parallel} = \frac{1}{1 + \varepsilon_1/\varepsilon_2} \rightarrow \sin\theta_{B_\parallel} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}} \qquad (9)$$

or

$$\tan \theta_{B_{\parallel}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \tag{10}$$

showing that there is a Brewster angle for any combination of ε_1 and ε_2.

Perpendicular Polarization

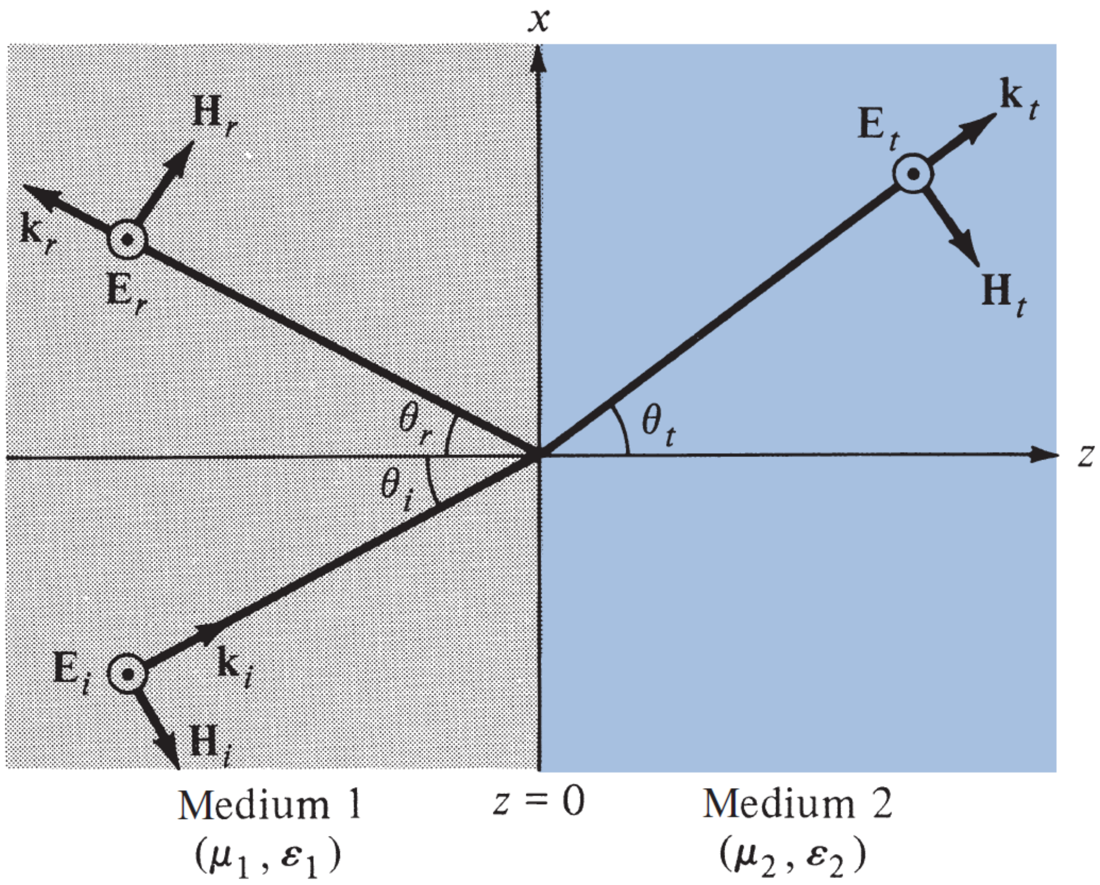


Figure 2: Oblique incidence with **E** perpendicular to the plane of incidence.

When the E field is perpendicular to the plane of incidence (the *xz*-plane) as shown in Figure 2, we have perpendicular polarization. This may also be viewed as the case in which the **H** field is parallel to the plane of incidence.

The incident and reflected fields in medium 1 are given by

$$\mathbf{E}_{is} = E_{io} e^{-j\beta_1 (x \sin\theta_i + z\cos\theta_i)} \mathbf{a}_y \quad (11a)$$

$$\mathbf{H}_{is} = \frac{E_{io}}{\eta_1} (-\cos\theta_i \, \mathbf{a}_x + \sin\theta_i \, \mathbf{a}_z) \, e^{-j\beta_1 (x \sin\theta_i + z\cos\theta_i)} \quad (11b)$$

$$\mathbf{E}_{rs} = E_{ro} e^{-j\beta_1 (x \sin\theta_r - z\cos\theta_r)} \, \mathbf{a}_y \quad (12a)$$

$$\mathbf{H}_{rs} = \frac{E_{ro}}{\eta_1} (\cos\theta_r \, \mathbf{a}_x + \sin\theta_r \, \mathbf{a}_z) \, e^{-j\beta_1 (x \sin\theta_r - z\cos\theta_r)} \quad (12b)$$

while the transmitted fields in medium 2 are given by

$$\mathbf{E}_{ts} = E_{to} e^{-j\beta_2(x \sin\theta_t + z\cos\theta_t)}\,\mathbf{a}_y \tag{13a}$$

$$\mathbf{H}_{ts} = \frac{E_{to}}{\eta_2}\left(-\cos\theta_t\,\mathbf{a}_x + \sin\theta_t\,\mathbf{a}_z\right) e^{-j\beta_2(x \sin\theta_t + z\cos\theta_t)} \tag{13b}$$

Notice that in defining the field components in eqs. (11) to (13), Maxwell's equations are always satisfied. Again, requiring that the tangential components of **E** and **H** be continuous at $z = 0$ and setting θ_r equal to θ_i, we get

$$E_{io} + E_{ro} = E_{to} \tag{14a}$$

$$\frac{1}{\eta_1}(E_{io} - E_{ro})\cos\theta_i = \frac{1}{\eta_2}E_{to}\cos\theta_t \tag{14b}$$

Expressing E_{ro} and E_{to} in terms of E_{io} leads to

$$\Gamma_{\perp} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \tag{15a}$$

or

$$E_{ro} = \Gamma_{\perp} E_{io} \tag{15b}$$

and

$$\tau_{\perp} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \tag{16a}$$

or

$$E_{to} = \tau_{\perp} E_{io} \tag{16b}$$

which are the *Fresnel's equations* for perpendicular polarization $\Gamma_\perp$ and $\tau_\perp$ are known as Fresnel coefficients. From eqs. (15) and (16), it is easy to show that

$$1 + \Gamma_\perp = \tau_\perp \tag{17}$$

which is similar to the equation obtained for normal incidence. Also, when $\theta_i = \theta_t = 0$, eqs. (15) and (16) reduces to the equations obtained for normal incidence.

For no reflection, $\Gamma_\perp = 0$ (or $E_r = 0$). This is the same as the case of total transmission ($\tau_\perp = 1$). By replacing θ_i with the corresponding Brewster angle $\theta_{B_\perp}$, we obtain

$$\eta_2 \cos\theta_{B_\perp} = \eta_1 \cos\theta_t$$

$$\eta_2^2(1 - \sin^2\theta_{B_\perp}) = \eta_1^2(1 - \sin^2\theta_t)$$

Incorporating the Snell's law for refraction yields

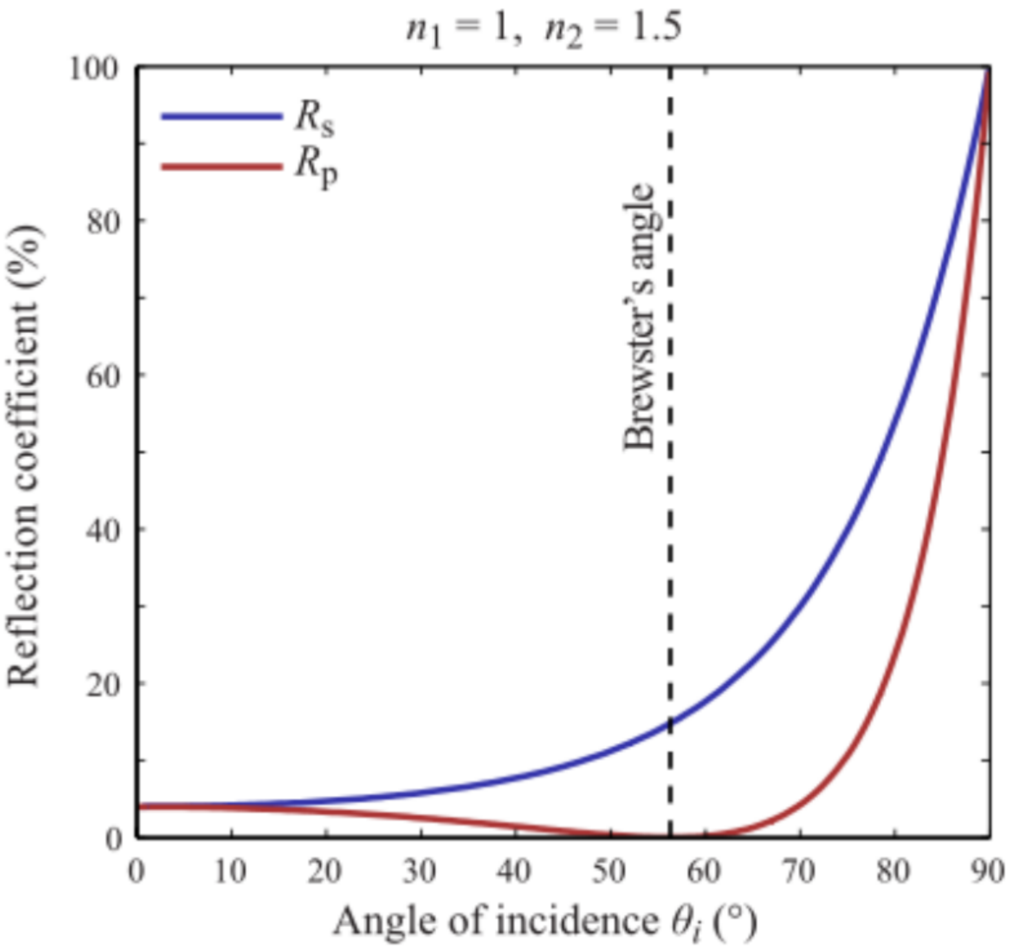
$$\sin^2 \theta_{B_\perp} = \frac{1 - \mu_1\varepsilon_2/\mu_2\varepsilon_1}{1 - (\mu_1/\mu_2)^2} \tag{18}$$

Note that for nonmagnetic media ($\mu_1 = \mu_2 = \mu_0$), $\sin^2 \theta_{B_\perp} \rightarrow \infty$ in eq. (18), so $\theta_{B_\perp}$ does not exist because the sine of an angle is never greater than unity. Also if $\mu_1 \neq \mu_2$ and $\varepsilon_1 = \varepsilon_2$, eq. (18) reduces to

$$\sin \theta_{B_\perp} = \sqrt{\frac{\mu_2}{\mu_1 + \mu_2}} \quad \text{or} \quad \tan \theta_{B_\perp} = \sqrt{\frac{\mu_2}{\mu_1}} \tag{19}$$

Although this situation is theoretically possible, it rarely occurs in practice.

❖ **Example:** 10.11



- **Q-1:** Define S-polarization, P-polarization and Brewster's angle?
- **Q-2:** Mathematically describe the differences in the appearance of reflection characteristics for S-polarized and P-polarized electromagnetic waves in the figure mentioned above.
- **Q-3:** Determine the Brewster's angle for the P-polarized case.

References

1. "Elements of Electromagnetics" - Matthew N. O. Sadiku 7th Edition (Chapter 10)