Chapter-7:
IP Traffic Engineering

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What is IP traffic?

- Traffic generated for different applications such as web, email etc, as seen at the IP-layer level as datagrams
  – May or may not classify by different applications
- Traffic volume: aggregated data rate for different services
  – Measured in packets per sec (pps), or bits per sec (bps)

Traffic data rate = Packets per sec x Average packet size

General relation on delay performance

- On a single link:

  Delay = F(traffic volume, capacity)

- In a network, routing is taken into consideration:

  Delay = F(traffic volume, capacity, routing)

- Also, sometimes controls need to be introduced as well; then,

  Delay = F(traffic volume, capacity, routing, control)

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Average Delay in a single-link

- For average Possion arrival rate λ, service rate μ ($\lambda < \mu$), the average delay (for M/M/1) is:

$$\tau = \frac{1}{\mu - \lambda}.$$

- If k is average packet size, then traffic volume h = $\kappa\ \lambda$, and capacity, c = $\kappa\ \mu$
- Thus,

$$\tau = \frac{\kappa}{\kappa(\mu - \lambda)} = \frac{\kappa}{c - h}.$$

This relation can be rewritten as:

$$\frac{\tau}{\kappa} = \frac{1}{c - h}.$$

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IP Traffic and delay

- IP traffic, however, does not follow Poisson. It's observed to self-similar.
  - No simple/nice result for self-similar traffic
  - Delay with self-similar traffic is known to be higher than Poisson traffic

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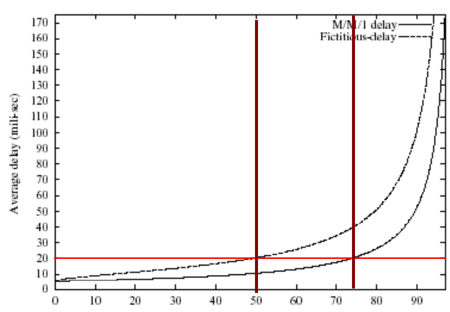
Poisson and non-Poisson Traffic



Means from a traffic engineering point of view, use lower utilization for an acceptable delay

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Non-stationary traffic

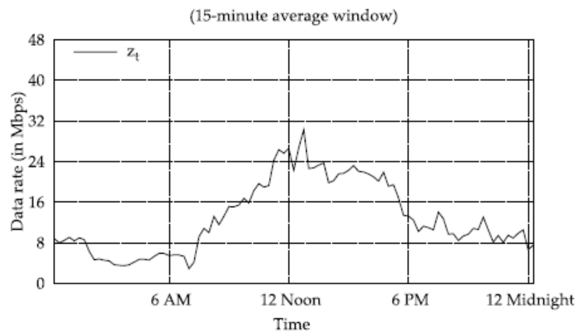

FIGURE 7.2 Traffic data rate over a 24-hour period.

Another reason to use lower utilization for an acceptable delay to allow for traffic variation

TCP Throughput, Bandwidth-delay product

- A useful result (good approximation for low loss environment)

$$\text{TCP throughput} = \frac{1.22\,S}{\mathcal{R}TT \times \sqrt{q}}.$$

  - S:= Maximum Segment Sizxe, RTT := Round trip time, and q := average packet loss probability
- Average packet loss depends on a number of factors (congestion, bit error rate etc)
- Congestion means a router may not have

Buffer Size at routers

- Typical rule for buffer size, *W*:

$$W = c \times \mathcal{RTT}.$$

- (*c* := Bandwidth rate)
- This has been a good rule of thumb for many years
- In a ultra high speed environment, need a fresh look
  - Consider, c = 40 Gbps, RTT = 250 ms
  - Then, W = 1.25 Gigabytes ! (this needs to be cache size)
  - Need a fresh look on buffer size (jury is still out there!)
- Note: buffer size is not directly part of IP traffic engineering as a router buffer size is carved in by router vendor based on interfaces etc; a network provider can inquire about, but has no control over it.
  - Important to know for understanding traffic engineering impact

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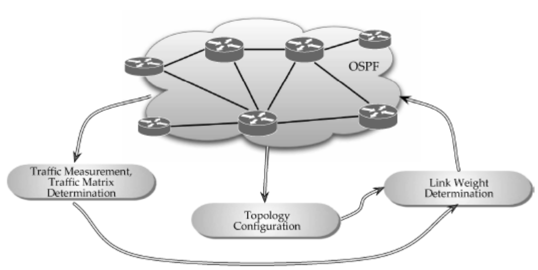
IP traffic engineering framework



- Provide acceptable delay to the network for a given traffic load by 'engineering' the network
  - Off-line link weight computation, which is fed back to the network so that shortest path routing use it
  - Need a goal/objective for traffic engineering

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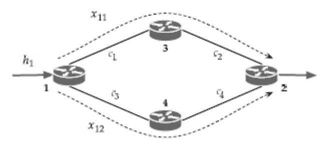
Network flow example

- Load balancing case:

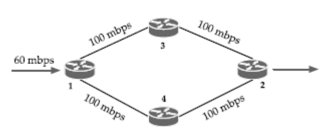
$$\begin{aligned} &\textit{minimize}_{\{x\}} && F = \max\left\{\frac{x_{11}}{c_1}, \frac{x_{11}}{c_2}, \frac{x_{12}}{c_3}, \frac{x_{12}}{c_4},\right\} \\ &\textit{subject to} && x_{11} + x_{12} = h_1 \\ & && x_{11} \le c_1, \quad x_{11} \le c_2, \quad x_{12} \le c_3, \quad x_{12} \le c_4 \\ & && x_{11} \ge 0, \quad x_{12} \ge 0. \end{aligned}$$



- Rewrite as:

$$\begin{aligned} &\textit{minimize}_{\{x,r\}} && F = r \\ &\textit{subject to} && x_{11} + x_{12} = h_1 \\ & && x_{11} \le c_1\, r, \quad x_{11} \le c_2\, r, \quad x_{12} \le c_3\, r, \quad x_{12} \le c_4\, r \\ & && x_{11} \ge 0, \quad x_{12} \ge 0. \end{aligned}$$

$$\begin{aligned} &\textit{minimize}_{\{x,r\}} && F = r \\ &\textit{subject to} && x_{11} + x_{12} = 60 \\ & && x_{11} \le 100\, r \\ & && x_{12} \le 100\, r \\ & && x_{11}, x_{12} \ge 0. \end{aligned}$$



- Solution:

$$x_{11}^* = x_{12}^* = 30.$$

Shortest Path Routing and Network flow

- Flow is induced by the 'cost' (weight) of links:
  - Suppose $\boldsymbol{w}$ is the set of link weights (SPR)

$$
\begin{aligned}
&\textit{minimize}_{\{w,r\}} && F = r \\
&\textit{subject to} && x_{11}(\boldsymbol{w}) + x_{12}(\boldsymbol{w}) = h_1 \\
& && x_{11}(\boldsymbol{w}) \le c_1 r, \quad x_{11}(\boldsymbol{w}) \le c_2 r, \\
& && x_{12}(\boldsymbol{w}) \le c_3 r, \quad x_{12}(\boldsymbol{w}) \le c_4 r \\
& && x_{11}(\boldsymbol{w}) \ge 0, \quad x_{12}(\boldsymbol{w}) \ge 0 \\
& && w_1, w_2, w_3, w_4 \in \mathcal{W}.
\end{aligned}
$$

Compare with ("pure" Netflow)

$$
\begin{aligned}
&\textit{minimize}_{\{x,r\}} && F = r \\
&\textit{subject to} && x_{11} + x_{12} = h_1 \\
& && x_{11} \le c_1 r, \quad x_{11} \le c_2 r, \quad x_{12} \le c_3 r, \quad x_{12} \le c_4 r \\
& && x_{11} \ge 0, \quad x_{12} \ge 0.
\end{aligned}
$$

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General observation

- At optimility:

$$F^*_{\text{netflow}} \le F^*_{\text{SPR}}.$$

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- Two 'classical' rules for picking link weights:

Rule-1: Choose the link weights to be based on hop count, to be referred to as a *hop-based* metric

Rule-2: Choose the link weights to be based on the inverse of the link speed, to be referred to as an *inverse-of-the-link speed* metric.

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Link Weight Determination Model with load balancing:

- For a general network, the problem can be formulated as:

$$\begin{array}{lll}
\textit{minimize}_{\{w,r\}} & F = r & \\
\textit{subject to} & \sum_{p=1}^{P_k} x_{kp}(\boldsymbol{w}) = h_k, & k = 1, 2, \ldots, K \\
& \sum_{k=1}^{K} \sum_{p=1}^{P_k} \delta_{kp\ell}\, x_{kp}(\boldsymbol{w}) = y_\ell, & \ell = 1, 2, \ldots, L \\
& y_\ell \le c_\ell\, r, & \ell = 1, 2, \ldots, L \\
& w_1, w_2, \ldots, w_L \in \mathcal{W} & \\
& x_{kp}(\boldsymbol{w}) \ge 0, & p = 1, 2, \ldots, P_k, \quad k = 1, 2, \ldots, K \\
& y_\ell \ge 0, & \ell = 1, 2, \ldots, L \\
& r \ge 0. &
\end{array} \qquad (7.5.1)$$

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Its relaxed version (without link weights as unknown)

$$
\begin{aligned}
&\textit{minimize}_{\{x,y,r\}} && F = r \\
&\textit{subject to} && \sum_{p=1}^{P_k} x_{kp} = h_k, && k = 1, 2, \ldots, K \\
& && \sum_{k=1}^{K} \sum_{p=1}^{P_k} \delta_{kp\ell} x_{kp} = y_\ell, && \ell = 1, 2, \ldots, L \\
& && y_\ell \le c_\ell r, && \ell = 1, 2, \ldots, L \\
& && x_{kp} \ge 0, && p = 1, 2, \ldots, P_k, \quad k = 1, 2, \ldots, K \\
& && y_\ell \ge 0, && \ell = 1, 2, \ldots, L \\
& && r \ge 0.
\end{aligned}
\qquad (7.5.2)
$$

Note:

$$F^*_{\text{MCNF}} \le F^*_{\text{MCSPRF}}.$$

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Notations

TABLE 7.1 Summary of notation used in MCNF and MCSPRF formulations.

Notation	Explanation
K	Number of demand pairs with positive demand volume
L	Number of links
h_k	Demand volume of demand index $k = 1, 2, \ldots, K$
c_ℓ	Capacity of link $\ell = 1, 2, \ldots, L$
P_k	Number of candidate paths for demand k, $k = 1, 2, \ldots, K$
$\delta_{kp\ell}$	Link-path indicator, set to 1 if path p for demand pair k uses the link ℓ; 0, otherwise
ξ_{kp}	Unit cost of flow on path p for demand k
$\hat{\xi}_\ell$	Unit cost of flow on link ℓ
w_ℓ	Link weight for link $\ell = 1, 2, \ldots, L$
$x_{kp}(\boldsymbol{w})$	Flow amount on path p for demand k for given link weight system $\boldsymbol{w}$
x_{kp}	Flow amount on path p for demand k
y_ℓ	Link flow variable for link ℓ
r	maximum link utilization variable
*	Use as a superscript with a variable to indicate optimal solution, e.g., x^*_{kp}

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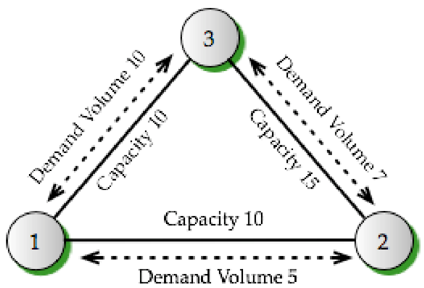
Remember from Chapter-4

Minimize 2 x12 + x132 + 2 x13 + x123 + 2 x23 + x213
subject to
d12: x12 + x132 = 5
d13: x13 + x123 = 10
d23: x23 + x213 = 7
c12: x12 + x123 + x213 <= 10
c13: x132 + x13 + x213 <= 10
c23: x132 + x123 + x23 <= 15
Bounds
0 <= x12
0 <= x132
0 <= x13
0 <= x123
0 <= x23
0 <= x213
End



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Review: Linear Programming duality

- Consider minimum cost routing (for simplicity)
  - Three-node problem from Chapter-4

$$\begin{aligned}
&\textit{minimize}_{\{x\}} \quad F = \xi_{11}x_{11} + \xi_{12}x_{12} + \xi_{21}x_{21} + \xi_{22}x_{22} + \xi_{31}x_{31} + \xi_{32}x_{32} \\
&\textit{subject to} \\
&x_{11} + x_{12} = h_1 \\
&x_{21} + x_{22} = h_2 \\
&x_{31} + x_{32} = h_3 \qquad (7.6.1) \\
&x_{11} + x_{22} + x_{32} \le c_1 \\
&x_{12} + x_{21} + x_{32} \le c_2 \\
&x_{12} + x_{22} + x_{31} \le c_3 \\
&x_{11},\ x_{12},\ x_{21},\ x_{22},\ x_{31},\ x_{32} \ge 0.
\end{aligned}$$

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- Rewrite as:

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$$\textit{minimize}_{\{x\}} \quad F = \hat{\xi}_1 x_{11} + (\hat{\xi}_2 + \hat{\xi}_3) x_{12} + \hat{\xi}_2 x_{21} + (\hat{\xi}_1 + \hat{\xi}_3) x_{22} + \hat{\xi}_3 x_{31} + (\hat{\xi}_1 + \hat{\xi}_3) x_{32}$$

subject to

$$\begin{array}{rrrrrrcrl} & & & & & & & & \text{(dual variables)} \\ x_{11} & +x_{12} & & & & & = & h_1 & (\nu_1) \\ & & x_{21} & +x_{22} & & & = & h_2 & (\nu_2) \\ & & & & x_{31} & +x_{32} & = & h_3 & (\nu_3) \\ -x_{11} & & & -x_{22} & & -x_{32} & \geq & -c_1 & (\pi_1) \\ & -x_{12} & -x_{21} & & & -x_{32} & \geq & -c_2 & (\pi_2) \\ & -x_{12} & & -x_{22} & -x_{31} & & \geq & -c_3 & (\pi_3) \end{array} \qquad (7.6.2)$$

$x_{11},\ x_{12},\ x_{21},\ x_{22},\ x_{31},\ x_{32} \geq 0.$

, $\xi_{11} = \hat{\xi}_1$, $\xi_{12} = \hat{\xi}_2 + \hat{\xi}_3$, $\xi_{21} = \hat{\xi}_2$, $\xi_{22} = \hat{\xi}_1 + \hat{\xi}_3$, $\xi_{31} = \hat{\xi}_3$, $\xi_{32} = \hat{\xi}_1 + \hat{\xi}_2$

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- Dual is:

$$\textit{maximize}_{\{\nu,\pi\}} \quad F_D = h_1\nu_1 + h_2\nu_2 + h_3\nu_3 - c_1\pi_1 - c_2\pi_2 - c_3\pi_3$$

subject to

$$\begin{array}{rrrrrrcl} \nu_1 & & & -\pi_1 & & & \leq & \hat{\xi}_1 \\ \nu_1 & & & & -\pi_2 & -\pi_3 & \leq & \hat{\xi}_2 + \hat{\xi}_3 \\ & \nu_2 & & & -\pi_2 & & \leq & \hat{\xi}_2 \\ & \nu_2 & & -\pi_1 & & -\pi_3 & \leq & \hat{\xi}_1 + \hat{\xi}_3 \\ & & \nu_3 & & & -\pi_3 & \leq & \hat{\xi}_3 \\ & & \nu_3 & -\pi_1 & -\pi_2 & & \leq & \hat{\xi}_1 + \hat{\xi}_2 \end{array} \qquad (7.6.3)$$

$\nu_1,\ \nu_2,\ \nu_3$ unrestricted

$\pi_1,\ \pi_2,\ \pi_3 \geq 0.$

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- Rewrite dual (7.6.3):

$$
\begin{aligned}
&\textit{maximize}_{\{\mathbf{v},\boldsymbol{\pi}\}} \quad F_D = h_1 v_1 + h_2 v_2 + h_3 v_3 - c_1\pi_1 - c_2\pi_2 - c_3\pi_3 \\
&\textit{subject to} \\
&\quad v_1 \le \hat{\xi}_1 + \pi_1 \\
&\quad v_1 \le (\hat{\xi}_2 + \pi_2) + (\hat{\xi}_3 + \pi_3) \\
&\quad v_2 \le \hat{\xi}_2 + \pi_2 \\
&\quad v_2 \le (\hat{\xi}_1 + \pi_1) + (\hat{\xi}_3 + \pi_3) \\
&\quad v_3 \le \hat{\xi}_3 + \pi_3 \\
&\quad v_3 \le (\hat{\xi}_1 + \pi_1) + (\hat{\xi}_2 + \pi_2) \\
&\quad v_1,\ v_2,\ v_3 \text{ unrestricted} \\
&\quad \pi_1,\ \pi_2,\ \pi_3 \ge 0.
\end{aligned}
$$

Note: dual constraints corresponds to paths for each demand v_i
for each demand the tighter requirement ("path cost") to be chosen

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N-node case:

- Minimum cost routing:

$$
\begin{aligned}
&\textit{minimize}_{\{x\}} && F = \sum_{k=1}^{K}\sum_{p=1}^{P_k} \xi_{kp} x_{kp} \\
&\textit{subject to} && \sum_{p=1}^{P_k} x_{kp} = h_k, && k = 1, 2, ..., K \\
& && \sum_{k=1}^{K}\sum_{p=1}^{P_k} \delta_{kp\ell} x_{kp} \le c_\ell, && \ell = 1, 2, ..., L \\
& && x_{kp} \ge 0, && p = 1, 2, ..., P_k, \quad k = 1, 2, ..., K.
\end{aligned}
\tag{7.6.5}
$$

- It's dual:

$$
\begin{aligned}
&\textit{maximize}_{\{\mathbf{v},\boldsymbol{\pi}\}} && F_D = \sum_{k=1}^{K} h_k v_k - \sum_{\ell=1}^{L} c_\ell \pi_\ell \\
&\textit{subject to} && v_k \le \sum_{\ell=1}^{L} \delta_{kp\ell}(\hat{\xi}_\ell + \pi_\ell), && p = 1, 2, ..., P_k, \quad k = 1, 2, ..., K \\
& && v_k \text{ unrestricted}, && k = 1, 2, ..., K \\
& && \pi_\ell \ge 0, && \ell = 1, 2, ..., L.
\end{aligned}
\tag{7.6.10}
$$

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Result from duality in determining link weights

Result 7.1. *For the MCNF problem given by Eq. (7.6.8) and its corresponding dual, Eq. (7.6.10), the commodity cost, v_k^*, is the shortest distance for demand k with respect to the link weight $w_\ell = \hat{\xi}_\ell + \pi_\ell^*$, and at optimality, every path for demand k that carries a positive flow must be a shortest path with respect to the link cost system given by*

$$w_\ell = \hat{\xi}_\ell + \pi_\ell^* \qquad (7.6.14)$$

for $\ell = 1, 2, ..., L$.

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For minimum cost Shortest path routing problem:

$$\begin{array}{lll} \text{minimize}_{\{w\}} & F = \sum_{k=1}^{K} \sum_{p=1}^{P_k} \xi_{kp} x_{kp}(\boldsymbol{w}) & \\ \text{subject to} & \sum_{p=1}^{P_k} x_{kp}(\boldsymbol{w}) = h_k, & k = 1, 2, ..., K \\ & \sum_{k=1}^{K} \sum_{p=1}^{P_k} \delta_{kp\ell} x_{kp}(\boldsymbol{w}) \le c_\ell, & \ell = 1, 2, ..., L \\ & w_1, w_2, ..., w_L \in \mathcal{W} & \\ & x_{kp}(\boldsymbol{w}) \ge 0, & p = 1, 2, ..., P_k, \quad k = 1, 2, ..., K. \end{array} \qquad (7.6.6)$$

- Use Result 7.1 to determine link weights
  - Means solve the dual problem first
    - For small problems, CPLEX readily gives the dual solution
    - For large problems, specialized algorithm needed

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Load Balancing case

- Minimize maximum utilization

$$\begin{array}{llll} \text{minimize}_{\{x,r\}} & F = r & & \\ \text{subject to} & \sum_{p=1}^{P_k} x_{kp} = h_k, & k = 1, 2, ..., K & (\nu_k) \\ & -\sum_{k=1}^{K}\sum_{p=1}^{P_k} \delta_{kp\ell}\, x_{kp} + c_\ell\, r \ge 0, & \ell = 1, 2, ..., L & (\pi_\ell) \\ & x_{kp} \ge 0, & p = 1, 2, ..., P_k, & \\ & & k = 1, 2, ..., K. & \\ & r \ge 0. & & \end{array} \quad (7.6.15)$$

- It's dual·

$$\begin{array}{llll} \text{maximize}_{\{\nu,\pi\}} & F = \sum_{k=1}^{K} h_k \nu_k & & \\ \text{subject to} & \nu_k - \sum_{\ell=1}^{L} \delta_{kp\ell}\pi_\ell \le 0, & p = 1, 2, ..., P_k, & k = 1, 2, ..., K \\ & \sum_{\ell=1}^{L} c_\ell \pi_\ell \le 1 & & \\ & \nu_k \text{ unrestricted}, & k = 1, 2, ..., K & \\ & \pi_\ell \ge 0, & \ell = 1, 2, ..., L. & \end{array} \quad (7.6.16)$$

- Rewrite dual (7.6.16):

$$\begin{array}{llll} \text{maximize}_{\{\nu,\pi\}} & F = \sum_{k=1}^{K} h_k \nu_k & & \\ \text{subject to} & \nu_k \le \sum_{\ell=1}^{L} \delta_{kp\ell}\pi_\ell, & p = 1, 2, ..., P_k, & k = 1, 2, ..., K \\ & \sum_{\ell=1}^{L} c_\ell \pi_\ell \le 1 & & \\ & \nu_k \text{ unrestricted}, & k = 1, 2, ..., K & \\ & \pi_\ell \ge 0, & \ell = 1, 2, ..., L. & \end{array} \quad (7.6.17)$$

Result 7.2. *For MCNF Formulation (7.6.15) and its corresponding dual given by Eq. (7.6.17), the commodity cost, ν_k^*, is the shortest distance for demand k with respect to link weight $w_\ell = \pi_\ell^*$, and at optimality, every path for demand k that carries a positive flow must be a shortest path with respect to the link cost system given by*

$$w_\ell = \pi_\ell^* \quad (7.6.19)$$

for $\ell = 1, 2, ..., L$ where $\sum_{\ell=1}^{L} c_\ell \pi_\ell^* = 1$.

- Compare the dual solution-based link weights between minimum cost routing and load balancing!

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How about a composite objective

$$
\begin{aligned}
\textit{minimize}_{\{x,r\}} \quad & F = \alpha \sum_{k=1}^{K} \sum_{p=1}^{P_k} \Big(\sum_{\ell=1}^{L} \hat{\xi}_\ell \delta_{kp\ell} \Big) x_{kp} + \beta r \\
\textit{subject to} \quad & \sum_{p=1}^{P_k} x_{kp} = h_k, \qquad k = 1, 2, ..., K \qquad (v_k) \\
& - \sum_{k=1}^{K} \sum_{p=1}^{P_k} \delta_{kp\ell} x_{kp} + c_\ell r \geq 0, \qquad \ell = 1, 2, ..., L \qquad (\pi_\ell) \\
& x_{kp} \geq 0, \qquad p = 1, 2, ..., P_k, \quad k = 1, 2, ..., K. \\
& r \geq 0.
\end{aligned}
\tag{7.6.20}
$$

Result 7.3. *For MCNF Formulation (7.6.20) and its corresponding dual, Eq. (7.6.22), the commodity cost, v_k^*, is the shortest distance for demand k with respect to link weight $w_\ell = \alpha\hat{\xi}_\ell + \pi_\ell^*$, and at optimality, every path for demand k that carries a positive flow must be a shortest path with respect to the link cost system given by*

$$w_\ell = \alpha\hat{\xi}_\ell + \pi_\ell^* \tag{7.6.23}$$

for $\ell = 1, 2, ..., L$ *where* $\sum_{\ell=1}^{L} c_\ell \pi_\ell^* = \beta$.

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Consider Average Delay Function

$$
\begin{array}{llll}
\textit{minimize}_{\{x,y\}} & F = \sum_{\ell=1}^{L} \frac{y_\ell}{c_\ell - y_\ell} & & \\
\textit{subject to} & \sum_{p=1}^{P_k} x_{kp} = h_k, & k = 1, 2, \dots, K & \\
& \sum_{k=1}^{K} \sum_{p=1}^{P_k} \delta_{kp\ell} x_{kp} = y_\ell, & \ell = 1, 2, \dots, L & \\
& y_\ell \le c_\ell, & \ell = 1, 2, \dots, L & \\
& x_{kp} \ge 0, & p = 1, 2, \dots, P_k, & k = 1, 2, \dots, K
\end{array}
\qquad (7.6.24)
$$

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Approximate with a piece-wise linear function

- Fortz-Thorup Function:

$$
\phi(y; c) = \begin{cases}
y & \text{for } 0 \le \frac{y}{c} < \frac{1}{3} \\
3y - \frac{2}{3}c & \text{for } \frac{1}{3} \le \frac{y}{c} < \frac{2}{3} \\
10y - \frac{16}{3}c & \text{for } \frac{2}{3} \le \frac{y}{c} < \frac{9}{10} \\
70y - \frac{178}{3}c & \text{for } \frac{9}{10} \le \frac{y}{c} < 1 \\
500y - \frac{1468}{3}c & \text{for } 1 \le \frac{y}{c} < \frac{11}{10} \\
5000y - \frac{16318}{3}c & \text{for } \frac{11}{10} \le \frac{y}{c} < \infty.
\end{cases}
$$

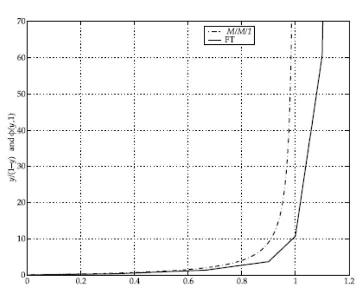

FIGURE 7.9 The Fortz–Thorup function and the load latency function (when $c = 1$).

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- The problem looks like:

$$\begin{aligned}
&minimize_{\{x,y\}} \quad F=\sum_{\ell=1}^{L}\frac{\phi(y_\ell;c_\ell)}{c_\ell}\\
&subject\ to \quad \sum_{p=1}^{P_k} x_{kp}=h_k, \qquad k=1,2,...,K\\
&\qquad \sum_{k=1}^{K}\sum_{p=1}^{P_k}\delta_{kp\ell}x_{kp}=y_\ell, \qquad \ell=1,2,...,L\\
&\qquad x_{kp}\geq 0, \qquad p=1,2,...,P_k, \quad k=1,2,...,K\\
&\qquad y_\ell\geq 0, \qquad \ell=1,2,...,L.
\end{aligned} \tag{7.6.26}$$

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$$\begin{aligned}
&minimize_{\{x,y,z\}} \quad F=\sum_{\ell=1}^{L}\frac{z_\ell}{c_\ell}\\
&subject\ to \quad \sum_{p=1}^{P_k} x_{kp}=h_k, \qquad k=1,2,...,K\\
&\qquad \sum_{k=1}^{K}\sum_{p=1}^{P_k}\delta_{kp\ell}x_{kp}=y_\ell, \qquad \ell=1,2,...,L\\
&\qquad z_\ell\geq y_\ell, \qquad \ell=1,2,...,L\\
&\qquad z_\ell\geq 3y_\ell-\tfrac{2}{3}c_\ell, \qquad \ell=1,2,...,L\\
&\qquad z_\ell\geq 10y_\ell-\tfrac{16}{3}c_\ell, \qquad \ell=1,2,...,L\\
&\qquad z_\ell\geq 70y_\ell-\tfrac{178}{3}c_\ell, \qquad \ell=1,2,...,L\\
&\qquad z_\ell\geq 500y_\ell-\tfrac{1468}{3}c_\ell, \qquad \ell=1,2,...,L\\
&\qquad z_\ell\geq 5000y_\ell-\tfrac{16318}{3}c_\ell, \qquad \ell=1,2,...,L\\
&\qquad x_{kp}\geq 0, \qquad p=1,2,...,P_k, \quad k=1,2,...,K\\
&\qquad y_\ell\geq 0, z_\ell\geq 0, \qquad \ell=1,2,...,L.
\end{aligned} \tag{7.6.27}$$

In compact form (for terms with z_l):

$$\begin{aligned}
&minimize_{\{x,y,z\}} \quad F=\sum_{\ell=1}^{L}\frac{z_\ell}{c_\ell}\\
&subject\ to \quad \sum_{p=1}^{P_k} x_{kp}=h_k, \qquad k=1,2,...,K\\
&\qquad \sum_{k=1}^{K}\sum_{p=1}^{P_k}\delta_{kp\ell}x_{kp}=y_\ell, \qquad \ell=1,2,...,L\\
&\qquad z_\ell\geq a_i y_\ell-b_i c_\ell, \qquad i=1,2,...,I, \ell=1,2,...,L\\
&\qquad x_{kp}\geq 0, \qquad p=1,2,...,P_k, \quad k=1,2,...,K\\
&\qquad y_\ell\geq 0, z_\ell\geq 0, \qquad \ell=1,2,...,L.
\end{aligned} \tag{7.6.28}$$

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Dual is:

$$
\begin{aligned}
&\text{maximize}_{\{v,\pi,\gamma\}} && \sum_{k=1}^{K} h_k v_k - \sum_{\ell=1}^{L}\sum_{i=1}^{I} b_i c_\ell \gamma_{\ell i} \\
&\text{subject to} && v_k \le \sum_{\ell=1}^{L} \delta_{kp}^{\ell} \pi_\ell, && p = 1,2,\dots,P_k, \quad k = 1,2,\dots,K \\
& && \sum_{i=1}^{I} a_i \gamma_{\ell i} \ge \pi_\ell, && \ell = 1,2,\dots,L \\
& && \sum_{i=1}^{I} \gamma_{\ell i} \le \frac{1}{c_\ell}, && \ell = 1,2,\dots,L \\
& && v_k \text{ unrestricted} \\
& && \pi_\ell, \gamma_{\ell i} \ge 0.
\end{aligned}
\tag{7.6.30}
$$

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$$
\begin{aligned}
&\text{minimize}_{\{x,y,z\}} && F = \sum_{\ell=1}^{L} \frac{z_\ell}{c_\ell} \\
&\text{subject to} && \sum_{p=1}^{P_k} x_{kp} = h_k, && k = 1,2,\dots,K \\
& && \sum_{k=1}^{K}\sum_{p=1}^{P_k} \delta_{kp\ell} x_{kp} = y_\ell, && \ell = 1,2,\dots,L \\
& && z_\ell \ge a_i y_\ell - b_i c_\ell, && i = 1,2,\dots,I,\ \ell = 1,2,\dots,L \\
& && x_{kp} \ge 0, && p = 1,2,\dots,P_k, \quad k = 1,2,\dots,K \\
& && y_\ell \ge 0, z_\ell \ge 0, && \ell = 1,2,\dots,L.
\end{aligned}
\tag{7.6.28}
$$

Result 7.4. *For each link $\ell = 1, 2, .., L$, assume that constraint $z_\ell \ge a_i y_\ell - b_i c_\ell$ for Problem Eq. (7.6.28) is binding for a unique i [denote by $i'(\ell)$] at optimality. Then an optimal link weight system is given by*

$$w_\ell^* = \pi_\ell^* = a_{i'(\ell)}, \quad \ell \in \mathcal{L}. \tag{7.6.31}$$

Uniqueness is, however, not always possible for every link; the general result then is as follows:

Result 7.5. *For each link $\ell = 1, 2, .., L$, constraint $z_\ell \ge a_i y_\ell - b_i c_\ell$ for Problem Eq. (7.6.28) can be binding for at most two consecutive i's [denote by $i'(\ell)$ and $i'(\ell) + 1$]. Furthermore, an optimal link weight system is given by*

$$w_\ell^* = \pi_\ell^* = a_{i'(\ell)} \gamma^*_{\ell,i'(\ell)} + a_{i'(\ell)+1} \gamma^*_{\ell,i'(\ell)+1}, \quad \ell \in \mathcal{L}, \tag{7.6.32}$$

where $\gamma^*_{\ell,i'(\ell)} + \gamma^*_{\ell,i'(\ell)+1} = 1/c_\ell$, $\gamma^*_{\ell,i'(\ell)}, \gamma^*_{\ell,i'(\ell)+1} \ge 0$.

Convex combination of adjacent slopes

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- If a software tool is used for solving the TE problem, then it's not necessary to write the dual since the dual solution is readily available.
  - In CPEX, retrieve the dual solution using 'display solution dual –'
- Be careful about writing the problem in a tool so that the signs of the constraints are properly represented!

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Illustration

X_{11}: Path 1-3-2
X_{12}: Path 1-4-2
X_{13}: Path 1-2

```
Minimize r
subject to
    nu_1:  x_11 + x_12 + x_13 = 60
    pi_1:  - x_11 + 100 r >= 0
    pi_2:  - x_11 + 100 r >= 0
    pi_3:  - x_12 + 100 r >= 0
    pi_4:  - x_12 + 100 r >= 0
    pi_5:  - x_13 + 100 r >= 0
Bounds
    0 <= x_11
    0 <= x_12
    0 <= x_13
End
```

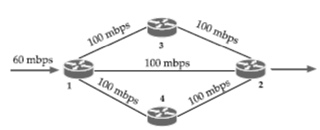

FIGURE 7.7 A four-node network example with five links.

Link ID:
1: 1-3
2: 3-2
3: 1-4
4: 4-2
5: 1-2

```
CPLEX> display solution dual -
Constraint Name          Dual Price
nu_1                     0.003333
pi_2                     0.003333
pi_4                     0.003333
pi_5                     0.003333
All other dual prices in the range 1–6 are zero.
```

By changing the coefficient of r to a large positive number, the pi's can be scaled up!

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Composite Objective

X_{11}: Path 1-3-2
X_{12}: Path 1-4-2
X_{13}: Path 1-2

Link ID:
1: 1-3
2: 3-2
3: 1-4
4: 4-2
5: 1-2

```
Minimize 2 x_11 + 2 x_12 + x_13 + 1000 r
subject to
    nu_1:  x_11 + x_12 + x_13 = 60
    pi_1:  - x_11 + 100 r >= 0
    pi_2:  - x_11 + 100 r >= 0
    pi_3:  - x_12 + 100 r >= 0
    pi_4:  - x_12 + 100 r >= 0
    pi_5:  - x_13 + 100 r >= 0
End
```

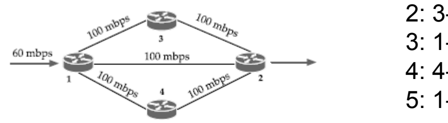

FIGURE 7.7 A four-node network example with five links.

On solving the above, we obtain dual solutions as follows:

```
CPLEX> display solution dual -
Constraint Name        Dual Price
nu_1                   5.000000
pi_2                   3.000000
pi_4                   3.000000
pi_5                   4.000000
All other dual prices in the range 1–6 are zero.
```

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Delay approximated by piece-wise linear function

```
Minimize z_1 + z_2 + z_3 + z_4 + z_5
subject to
    nu_1:  x_11 + x_12 + x_13 = 60
    pi_1:  - x_11 + y_1 >= 0
    pi_2:  - x_11 + y_2 >= 0
    pi_3:  - x_12 + y_3 >= 0
    pi_4:  - x_12 + y_4 >= 0
    pi_5:  - x_13 + y_5 >= 0
    gamma_1_1: z_1 - 1 y_1 >= - 0
    gamma_1_2: z_1 - 3 y_1 >= - 66.6667
    gamma_1_3: z_1 - 10 y_1 >= - 533.333
    gamma_1_4: z_1 - 70 y_1 >= - 5933.33
    gamma_1_5: z_1 - 500 y_1 >= - 48933.3
    gamma_1_6: z_1 - 5000 y_1 >= - 543933
    gamma_2_1: z_2 - 1 y_2 >= - 0
    gamma_2_2: z_2 - 3 y_2 >= - 66.6667
    gamma_2_3: z_2 - 10 y_2 >= - 533.333
    gamma_2_4: z_2 - 70 y_2 >= - 5933.33
    gamma_2_5: z_2 - 500 y_2 >= - 48933.3
    gamma_2_6: z_2 - 5000 y_2 >= - 543933
    gamma_3_1: z_3 - 1 y_3 >= - 0
    gamma_3_2: z_3 - 3 y_3 >= - 66.6667
    gamma_3_3: z_3 - 10 y_3 >= - 533.333
    gamma_3_4: z_3 - 70 y_3 >= - 5933.33
    gamma_3_5: z_3 - 500 y_3 >= - 48933.3
    gamma_3_6: z_3 - 5000 y_3 >= - 543933
    gamma_4_1: z_4 - 1 y_4 >= - 0
    gamma_4_2: z_4 - 3 y_4 >= - 66.6667
    gamma_4_3: z_4 - 10 y_4 >= - 533.333
    gamma_4_4: z_4 - 70 y_4 >= - 5933.33
    gamma_4_5: z_4 - 500 y_4 >= - 48933.3
    gamma_4_6: z_4 - 5000 y_4 >= - 543933
    gamma_5_1: z_5 - 1 y_5 >= - 0
    gamma_5_2: z_5 - 3 y_5 >= - 66.6667
    gamma_5_3: z_5 - 10 y_5 >= - 533.333
    gamma_5_4: z_5 - 70 y_5 >= - 5933.33
    gamma_5_5: z_5 - 500 y_5 >= - 48933.3
    gamma_5_6: z_5 - 5000 y_5 >= - 543933
End
```

```
CPLEX> display solution dual -
Constraint Name     Dual Price
nu_1                2.000000
pi_1                1.000000
pi_2                1.000000
pi_3                1.000000
pi_4                1.000000
pi_5                2.000000
gamma_1_1           1.000000
gamma_2_1           1.000000
gamma_3_1           1.000000
gamma_4_1           1.000000
gamma_5_1           0.500000
gamma_5_2           0.500000
All other dual prices in the range 1–36 are zero.
```

Note (compare to Result 7.5):

$$\pi_5^* = a_1\gamma_{51} + a_2\gamma_{52} = 1 \times 0.5 + 3 \times 0.5 = 2.$$

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- See Case Study-II (Page-231) for another illustration

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Using Specialized Algorithm for large problems:

TABLE 7.2 Results using the minimum cost routing objective for 100-node networks.

Nodal Degree (Number of Links)	ML	FU	FT	Solution Gap	F/I(NOL, FE)	Computing Time
2 (197)	0.79	0.25	1.22	0.6%	F(-)	44 sec
3 (294)	0.68	0.24	1.15	0.5%	F(-)	13 sec
4 (390)	0.48	0.18	1.04	0.6%	F(-)	60 sec
5 (485)	0.50	0.19	1.04	0.2%	F(-)	20 sec
6 (579)	0.49	0.17	1.02	0.4%	F(-)	23 sec

TABLE 7.3 Results using the composite objective function for 100-node networks.

Nodal Degree (Number of Links)	(α, β)	ML	FU	FT	Solution Gap	F/I(NOL, FE)	Computing Time
2 (197)	(0.9, 32)	0.67	0.25	1.15	0.1%	F(-)	5 sec
3 (294)	(0.9, 11585)	0.66	0.24	1.15	4.4%	F(-)	21 sec
4 (390)	(0.9, 2896)	0.37	0.18	1.00	0.7%	F(-)	19 sec
5 (485)	(0.9, 2896)	0.36	0.19	1.00	0.5%	F(-)	16 sec
6 (579)	(0.9, 16)	0.39	0.17	1.00	0.1%	F(-)	5 sec

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Summary

- IP traffic engineering
- How to find optimal link weights
- Only dual-based link weight determination is described here
  - Other approaches possible

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