Chapter-4: Network Flow Modeling & Optimization

Deep Medhi and Karthik Ramasamy

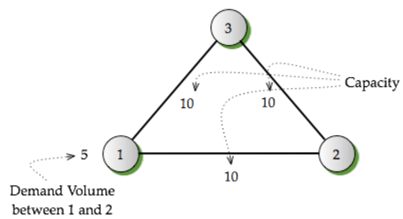
August 2007

http://www.NetworkRouting.net

Terminologies

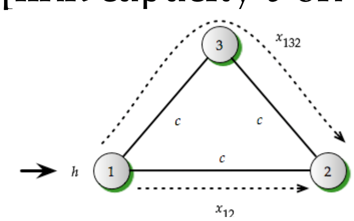
- Traffic volume/ demand volume
  - Different units for different purpose
    - Mbps, Gbps : data rate (both data networks and transport networks)
    - Erlang (telephone calls)
    - First to discuss in terms of pure numbers; later we'll use how the above units are used
- Demand pair, or node pair: between two points(nodes)
  - 1:2, 2:3
- Link: connects two points (nodes) directly
  - Bidirectional (1-2) or unidirectional (1->2)
- Flow: aggregation of traffic:
  - Link flow (for different pairs of demand)
- Capacitated Network: links have given capacity

A three-node illustration



Single-Commodity flow

- Consider just one demand 1:2 with volume *h* [link capacity *c* on all links]



$$x_{12} + x_{132} = h$$

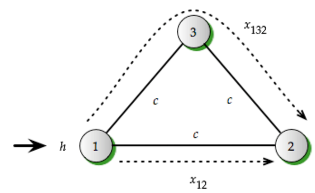
$$x_{12} \geq 0, \qquad x_{132} \geq 0.$$

$$x_{12} \leq c, \qquad x_{132} \leq c.$$

- Consider unit cost of paths
- Now we optimize cost of routing

$$\begin{aligned} &\textbf{\textit{minimize}}_{\{x_{12},x_{132}\}} && F = \xi_{12}x_{12} + \xi_{132}x_{132} \\ &\textbf{\textit{subject to}} && x_{12} + x_{132} = h \\ & && x_{12} \le c,\ x_{132} \le c \\ & && x_{12} \ge 0,\ x_{132} \ge 0. \end{aligned}$$

- Minimum cost routing
- Model (4.2.3)



Solution

- For different scenarios:

  $\xi_{12} = 1$, $\xi_{132} = 2$ (first path is cheaper than second part)
  - Optimal:

    $x_{12}^* = h$, $x_{132}^* = 0$ if $0 \le h \le 10$

    $x_{12}^* = 10$, $x_{132}^* = h - 10$, if $10 \le h \le 20$

  $\xi_{12} = 2$, $\xi_{132} = 1$ (second path is cheaper than first part)
  - Optimal:

    $x_{12}^* = 0$, $x_{132}^* = h$ if $0 \le h \le 10$

    $x_{12}^* = h\text{-}10$, $x_{132}^* = 10$, if $10 \le h \le 20$

Another objective: load balancing

- Minimize maximum load on a link (4.2.6):

$$
\begin{aligned}
&\textit{\textbf{minimize}}_{\{x\}} && F = \max\{\tfrac{x_{12}}{c},\ \tfrac{x_{132}}{c}\} \\
&\textit{\textbf{subject to}} && x_{12} + x_{132} = h \\
& && x_{12} \le c,\ x_{132} \le c \\
& && x_{12} \ge 0,\ x_{132} \ge 0.
\end{aligned}
$$

7

- In this case, it's optimal when the load (utilization) is equal

$$\frac{x_{12}^*}{c} = \frac{x_{132}^*}{c}$$

$$\frac{x_{12}^*}{c} = \frac{h - x_{12}^*}{c}$$

$$x_{12}^* = h/2.$$

- Thus, in this case, split equally
  - [Note: this work only in the case of two segments]

8

Another objective: minimize delay (4.2.9)

$$\begin{array}{ll} \textbf{\textit{minimize}}_{\{x\}} & F = \frac{x_{12}}{c - x_{12}} + \frac{2x_{132}}{c - x_{132}} \\ \textbf{\textit{subject to}} & x_{12} + x_{132} = h \\ & x_{12} \le c, \;\; x_{132} \le c \\ & x_{12} \ge 0, \;\; x_{132} \ge 0. \end{array}$$

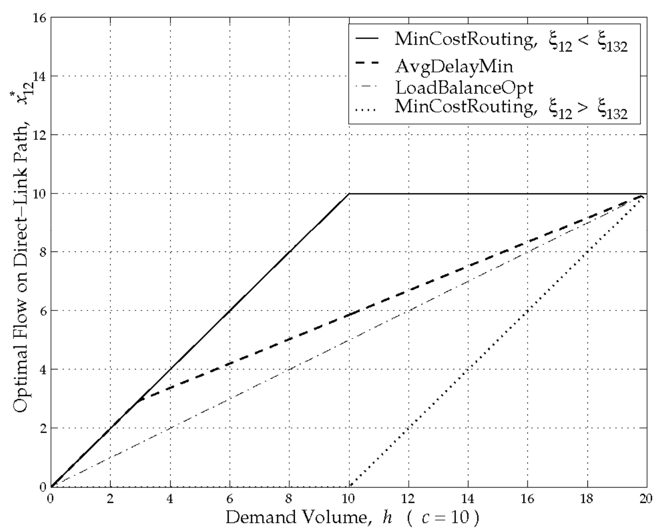
- Objective is non-linear; solution (after some math)

$$x_{12}^{*} = \min\{h, \;\; -h + 3c - 2\sqrt{2}c + \sqrt{2}h\}.$$

[first, reduce to one variable. Then differentiate with respect to the other Variable and set to zero. Finally, bounds are applied because the solution Must lie in the allowed region.]

9

Observations about solutions for different objectives



10

Figure 4.3

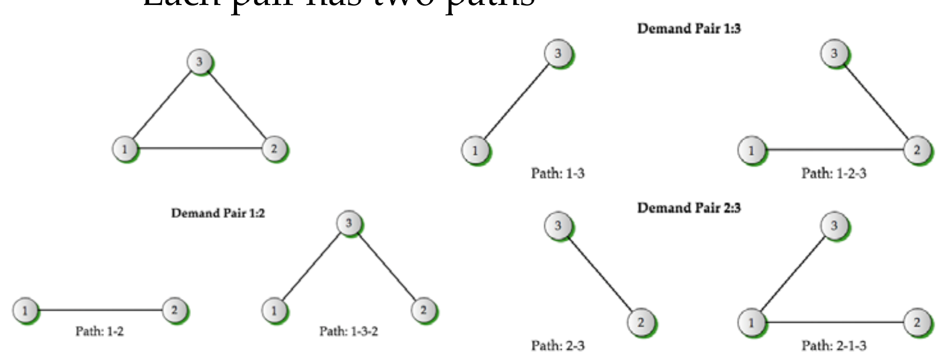
Summary from Figure 4.3

- For different objectives, the solution quality can be quite different
- However, at extreme ends (very low traffic or very high traffic compared to capacity), the solutions are quite similar

11

Now consider multiple demands: Multi-commodity flow

- 3-node network: all node pairs have traffic
  - Each pair has two paths



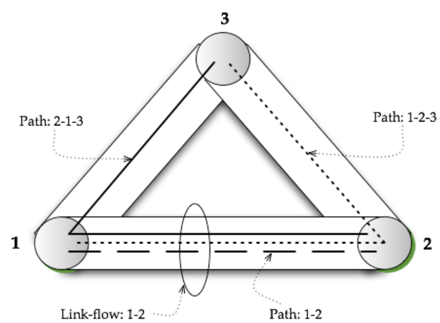
12

- Path flow for each node pair:

$$x_{12} + x_{132} = h_{12}.$$

$$x_{13} + x_{123} = h_{13}$$

$$x_{23} + x_{213} = h_{23}.$$



- Link flow:

$$x_{12} + x_{123} + x_{213} \le c_{12}.$$

$$x_{13} + x_{132} + x_{213} \le c_{13}$$

$$x_{23} + x_{132} + x_{123} \le c_{23}$$

13

Minimum cost routing optimization

- Model (4.3.4)

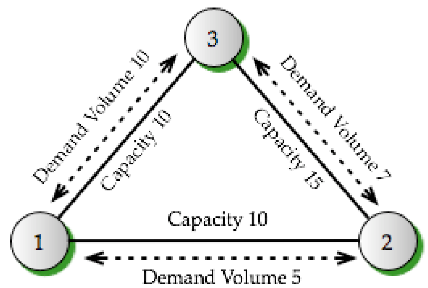
$$\begin{array}{ll}
\textbf{\textit{minimize}}_{\{x\}} & F = \xi_{12}x_{12} + \xi_{132}x_{132} + \xi_{13}x_{13} + \xi_{123}x_{123} + \xi_{23}x_{23} + \xi_{213}x_{213} \\
\textbf{\textit{subject to}} & x_{12} + x_{132} = h_{12} \\
& x_{13} + x_{123} = h_{13} \\
& x_{23} + x_{213} = h_{23} \\
& x_{12} + x_{123} + x_{213} \le c_{12} \\
& x_{13} + x_{132} + x_{213} \le c_{13} \\
& x_{23} + x_{132} + x_{123} \le c_{23} \\
& x_{12} \ge 0,\ x_{132} \ge 0,\ x_{13} \ge 0,\ x_{123} \ge 0,\ x_{23} \ge 0,\ x_{213} \ge 0.
\end{array}$$

14

Solving using CPLEX

- <Example 4.2>

```
Minimize 2 x12 +  x132 + 2 x13 +  x123 + 2 x23 +  x213
subject to
     d12:   x12 + x132 = 5
     d13:   x13 + x123 = 10
     d23:   x23 + x213 = 7
     c12:   x12 + x123 + x213 <= 10
     c13:   x132 + x13 + x213 <= 10
     c23:   x132 + x123 + x23 <= 15
Bounds
     0 <= x12
     0 <= x132
     0 <= x13
     0 <= x123
     0 <= x23
     0 <= x213
End
```



CPLEX output

```
CPLEX> display solution variables -
Variable Name          Solution Value
x12                    1.000000
x132                   4.000000
x13                    3.500000
x123                   6.500000
x23                    4.500000
x213                   2.500000
```

Thus, we have $x_{12}^* = 1, x_{132}^* = 4, x_{13}^* = 3.5, x_{123}^* = 6.5, x_{23}^* = 4.5, x_{213}^* = 2.5.$

What happens if the variables take only integer values (integer linear programming)

$$
\begin{aligned}
\textbf{\textit{minimize}}_{\{x\}} \quad & F = \xi_{12}x_{12} + \xi_{132}x_{132} + \xi_{13}x_{13} + \xi_{123}x_{123} + \xi_{23}x_{23} + \xi_{213}x_{213} \\
\textbf{\textit{subject to}} \quad & x_{12} + x_{132} = h_{12} \\
& x_{13} + x_{123} = h_{13} \\
& x_{23} + x_{213} = h_{23} \\
& x_{12} + x_{123} + x_{213} \le c_{12} \\
& x_{13} + x_{132} + x_{213} \le c_{13} \\
& x_{23} + x_{132} + x_{123} \le c_{23} \\
& x_{12} \ge 0,\ x_{132} \ge 0,\ x_{13} \ge 0,\ x_{123} \ge 0,\ x_{23} \ge 0,\ x_{213} \ge 0 \\
& \text{all } xs \text{ integer.}
\end{aligned}
\tag{4.3.5}
$$

17

Using CPLEX for integer linear programming (ILP)

```
Minimize 2 x12 +  x132 + 2 x13 +  x123 + 2 x23 +  x213
subject to
     d1:  x12 + x132 = 5
     d2:  x13 + x123 = 10
     d3:  x23 + x213 = 7
     c1:  x12 + x123 + x213 <= 10
     c2:  x132 + x13 + x213 <= 10
     c3:  x132 + x123 + x23 <= 15
Bounds
     0 <= x12 <= 10
     0 <= x132 <= 10
     0 <= x13 <= 10
     0 <= x123 <= 10
     0 <= x23 <= 10
     0 <= x213 <= 10
Integer
     x12 x132 x13 x123 x23 x213
End
```

Declare appropriate bound (default is 0 <= x <= 1)

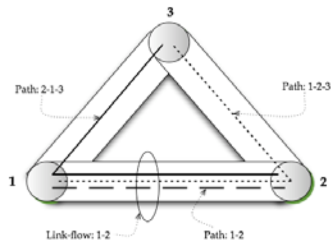
Declare Integer variables

Solution: $x_{12}^* = 1, x_{132}^* = 4, x_{13}^* = 4, x_{123}^* = 6, x_{23}^* = 5, x_{213}^* = 2$

18

Load balancing

- Minimize maximum Link utilization (4.3.8)



$$
\begin{aligned}
&\textbf{\textit{minimize}}_{\{x,y\}} && F = \max\left\{\frac{y_{12}}{c_{12}}, \frac{y_{13}}{c_{13}}, \frac{y_{23}}{c_{23}}\right\} \\
&\textbf{\textit{subject to}} && x_{12} + x_{132} = h_{12} \\
& && x_{13} + x_{123} = h_{13} \\
& && x_{23} + x_{213} = h_{23} \\
& && x_{12} + x_{123} + x_{213} = y_{12} \\
& && x_{13} + x_{132} + x_{213} = y_{13} \\
& && x_{23} + x_{132} + x_{123} = y_{23} \\
& && y_{12} \le c_{12},\ y_{13} \le c_{13},\ y_{23} \le c_{23} \\
& && x_{12} \ge 0,\ x_{132} \ge 0,\ x_{13} \ge 0,\ x_{123} \ge 0, x_{23} \ge 0,\ x_{213} \ge 0 \\
& && y_{12} \ge 0,\ y_{13} \ge 0,\ y_{23} \ge 0.
\end{aligned}
\qquad (4.3.8)
$$

Link flow

Objective is piece-wise non-linear

19

Transform objective to a linear equivalent case:

- Set

$$r = \max\left\{\frac{y_{12}}{c_{12}}, \frac{y_{13}}{c_{13}}, \frac{y_{23}}{c_{23}}\right\},$$

- Now,

$$r \ge \frac{y_{12}}{c_{12}}, \quad r \ge \frac{y_{13}}{c_{13}}, \quad r \ge \frac{y_{23}}{c_{23}}.$$

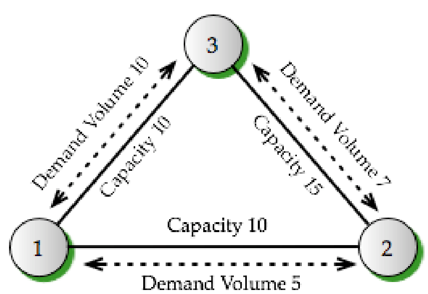
20

Transform (4.3.8) to the linear case: (4.3.12)

$$
\begin{aligned}
&\textbf{\textit{minimize}}_{\{x,y,r\}} && F = r \\
&\textbf{\textit{subject to}} && x_{12} + x_{132} = h_{12} \\
& && x_{13} + x_{123} = h_{13} \\
& && x_{23} + x_{213} = h_{23} \\
& && x_{12} + x_{123} + x_{213} = y_{12} \\
& && x_{13} + x_{132} + x_{213} = y_{13} \\
& && x_{23} + x_{132} + x_{123} = y_{23} \\
& && \cancel{y_{12} \le c_{12},\ y_{13} \le c_{13},\ y_{23} \le c_{23}} \quad \text{redundant} \\
& && y_{12} \le c_{12} r,\ y_{13} \le c_{13} r,\ y_{23} \le c_{23} r \\
& && x_{12} \ge 0,\ x_{132} \ge 0,\ x_{13} \ge 0,\ x_{123} \ge 0,\ x_{23} \ge 0,\ x_{213} \ge 0 \\
& && y_{12} \ge 0,\ y_{13} \ge 0,\ y_{23} \ge 0.
\end{aligned}
\qquad (4.3.12)
$$

21

Solution: load balancing



$$x^*_{12} = 5, x^*_{13} = 7.5, x^*_{123} = 2.5, x^*_{23} = 7$$

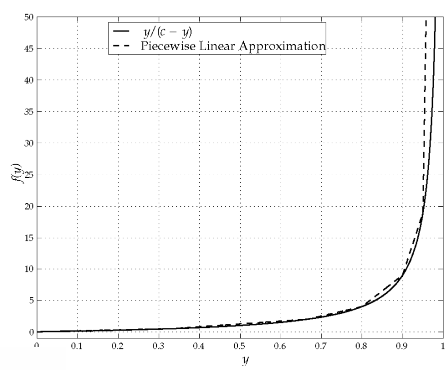
22

Average Delay case

$$
\begin{aligned}
\textbf{\textit{minimize}}_{\{x,y\}} \quad & F = \left(\frac{y_{12}}{c_{12}-y_{12}} + \frac{y_{13}}{c_{13}-y_{13}} + \frac{y_{23}}{c_{23}-y_{23}}\right) \\
\textbf{\textit{subject to}} \quad & x_{12} + x_{132} = h_{12} \\
& x_{13} + x_{123} = h_{13} \\
& x_{23} + x_{213} = h_{23} \\
& x_{12} + x_{123} + x_{213} = y_{12} \\
& x_{13} + x_{132} + x_{213} = y_{13} \\
& x_{23} + x_{132} + x_{123} = y_{23} \\
& y_{12} \le c_{12},\ y_{13} \le c_{13},\ y_{23} \le c_{23} \\
& x_{12} \ge 0,\ x_{132} \ge 0,\ x_{13} \ge 0,\ x_{123} \ge 0,\ x_{23} \ge 0,\ x_{213} \ge 0 \\
& y_{12} \ge 0,\ y_{13} \ge 0,\ y_{23} \ge 0.
\end{aligned}
\tag{4.3.14}
$$

23

Piece-wise linear approximation



- Approximate

$$f(y) = \frac{y}{c-y}, \quad \text{for } 0 \le y/c < 1$$

as:

$$
\tilde{f}(y) = \begin{cases}
\frac{3}{2}y, & \text{for } 0 \le \frac{y}{c} \le \frac{1}{3} \\
\frac{9}{2}y - c, & \text{for } \frac{1}{3} \le \frac{y}{c} < \frac{2}{3} \\
15\,y - 8c, & \text{for } \frac{2}{3} \le \frac{y}{c} < \frac{4}{5} \\
50\,y - 36c, & \text{for } \frac{4}{5} \le \frac{y}{c} < \frac{9}{10} \\
200\,y - 171c, & \text{for } \frac{9}{10} \le \frac{y}{c} < \frac{19}{20} \\
4000\,y - 3781c, & \text{for } \frac{y}{c} \ge \frac{19}{20}.
\end{cases}
$$

$$
\tilde{f}(y) = \max\{\tfrac{3}{2}y,\ \tfrac{9}{2}y - c,\ 15y - 8c,\ 50y - 36c,\ 200y - 171c,\ 4000y - 3781c\}, \quad \text{for } y \ge 0.
$$

24

Model (4.3.14) is approximated as Model (4.3.19)

$$
\begin{array}{lll}
\textbf{minimize}_{\{x,y,r\}} & F = \frac{1}{c_{12}} r_{12} + \frac{1}{c_{13}} r_{13} + \frac{1}{c_{23}} r_{23} & \\
\textbf{subject to} & x_{12} + x_{132} = h_{12} & \\
& x_{13} + x_{123} = h_{13} & \\
& x_{23} + x_{213} = h_{23} & \\
& x_{12} + x_{123} + x_{213} = y_{12} & \\
& x_{13} + x_{132} + x_{213} = y_{13} & \\
& x_{23} + x_{132} + x_{123} = y_{23} & \\
& r_{ij} \ge \frac{3}{2} y_{ij}, & (i,j) = (1,2), (1,3), (2,3) \\
& r_{ij} \ge \frac{9}{2} y_{ij} - c_{ij}, & (i,j) = (1,2), (1,3), (2,3) \\
& r_{ij} \ge 15 y_{ij} - 8 c_{ij}, & (i,j) = (1,2), (1,3), (2,3) \\
& r_{ij} \ge 50 y_{ij} - 36 c_{ij}, & (i,j) = (1,2), (1,3), (2,3) \\
& r_{ij} \ge 200 y_{ij} - 171 c_{ij}, & (i,j) = (1,2), (1,3), (2,3) \\
& r_{ij} \ge 4000 y_{ij} - 3781 c_{ij}, & (i,j) = (1,2), (1,3), (2,3) \\
& x_{12} \ge 0,\ x_{132} \ge 0,\ x_{13} \ge 0 & \\
& x_{123} \ge 0, x_{23} \ge 0,\ x_{213} \ge 0 & \\
& y_{12} \ge 0,\ y_{13} \ge 0,\ y_{23} \ge 0 & \\
& r_{12} \ge 0,\ r_{13} \ge 0,\ r_{23} \ge 0. &
\end{array}
$$

25

General Case: New notation

- Notation: renumbering (first with 3-node)

Pair	Index
1:2	1
1:3	2
2:3	3

Link	Index
1-2	1
1-3	2
2-3	3

- Now, (4.3.4) becomes following model (4.4.3):

$$
\begin{array}{llllllll}
\textbf{minimize}_{\{x\}} & \multicolumn{7}{l}{F = \xi_{11}x_{11} + \xi_{12}x_{12} + \xi_{21}x_{21} + \xi_{22}x_{22} + \xi_{31}x_{31} + \xi_{32}x_{32}} \\
\textbf{subject to} \\
x_{11} & +x_{12} & & & & & = & h_1 \\
& & x_{21} & +x_{22} & & & = & h_2 \\
& & & & x_{31} & +x_{32} & = & h_3 \\
x_{11} & & & +x_{22} & & +x_{32} & \le & c_1 \\
& x_{12} & +x_{21} & & & +x_{32} & \le & c_2 \\
& x_{12} & & +x_{22} & +x_{31} & & \le & c_3 \\
\multicolumn{8}{l}{x_{11},\ x_{12},\ x_{21},\ x_{22},\ x_{31},\ x_{32} \ge 0.}
\end{array}
$$

- Look at the relation: Link to Path

TABLE 4.1 Link-path incidence information.

Link\Path	$k=1, p=1$	$k=1, p=2$	$k=2, p=1$	$k=2, p=2$	$k=3, p=1$	$k=3, p=2$
1	1	0	0	1	0	1
2	0	1	1	0	0	1
3	0	1	0	1	1	0

- Use ‘delta’ notation:

$\delta_{kp\ell} = 1$ if path p for demand pair k uses the link ℓ; 0, otherwise.

Path 1 for demand pair 1 uses link 1 -> 1
Path 1 for demand pair 1 uses link 2 -> 0

27

Notations:

TABLE 4.2 Notation used in the link-path formulation.

Notation	Explanation
Given:	
K	Number of demand pairs with positive demand volume
L	Number of links
h_k	Demand volume of demand index $k = 1, 2, \ldots, K$
c_ℓ	Capacity of link $\ell = 1, 2, \ldots, L$
P_k	Number of candidate paths for demand k, $k = 1, 2, \ldots, K$
$\delta_{kp\ell}$	Link-path indicator, set to 1 if path p for demand pair k uses the link ℓ; 0, otherwise
ξ_{kp}	Nonnegative unit cost of flow on path p for demand k
Variables:	
x_{kp}	Flow amount on path p for demand k
y_ℓ	Link-flow variable for link ℓ

28

- Generic demand flow for *K* demands:

$$\sum_{p=1}^{P_k} x_{kp} = h_k, \quad k = 1, 2, \ldots, K.$$

- Link flow for each link (ℓ=1,2…,L):

$$\sum_{k=1}^{K}\sum_{p=1}^{P_k} \delta_{kp\ell} x_{kp} = y_\ell, \quad \ell = 1, 2, \ldots, L.$$

- Capacity constraint:

$$y_\ell = \sum_{k=1}^{K}\sum_{p=1}^{P_k} \delta_{kp\ell} x_{kp} \le c_\ell, \quad \ell = 1, 2, \ldots, L.$$

29

Minimum Cost Routing: General Case

- Link-Path formulation (4.4.7):

$$\textbf{\textit{minimize}}_{\{x\}} \quad F = \sum_{k=1}^{K}\sum_{p=1}^{P_k} \xi_{kp} x_{kp}$$

subject to

$$\sum_{p=1}^{P_k} x_{kp} = h_k, \qquad k = 1, 2, \ldots, K$$

$$\sum_{k=1}^{K}\sum_{p=1}^{P_k} \delta_{kp\ell} x_{kp} \le c_\ell, \qquad \ell = 1, 2, \ldots, L$$

$$x_{kp} \ge 0, \qquad p = 1, 2, \ldots, P_k,\ k = 1, 2, \ldots, K.$$

30

Some numbers

- Link-Path formulation

TABLE 4.3 Size of minimum cost routing problem (undirected network).

N	L	$\overline{P}_k$ (average)	Variables	Constraints	
				Demand	Capacity
5	7	4	40	10	7
10	30	7	315	45	30
50	200	10	12,250	1225	200

- Means: need to generate 'possible' paths
  - Use k-shortest path algorithm (or other means)

31

An important result

- Result 4.1
  - If (4.4.7) is feasible, then at most K+L flow variables are required to be nonzero at optimality

Illustration:

N = 50, L = 200, K = 1225

Of total 12,250 flow variables, at most K+L = 1,425 would need to be non-zero at optimality

Since ther are 1,225 demand pair, each pair must have at least one path that's non-zero, which means at most 200 demand pairs would have more than one non-zero path at optimality

32

Load balancing: general case

- (4.4.10):

$$
\begin{aligned}
&\textbf{\textit{minimize}}_{\{x,y,r\}} \quad F = r \\
&\textbf{\textit{subject to}} \\
&\quad \sum_{p=1}^{P_k} x_{kp} = h_k, && k = 1, 2, \ldots, K \\
&\quad \sum_{k=1}^{K} \sum_{p=1}^{P_k} \delta_{kp\ell}\, x_{kp} = y_\ell, && \ell = 1, 2, \ldots, L \\
&\quad y_\ell \le c_\ell r, && \ell = 1, 2, \ldots, L \\
&\quad x_{kp} \ge 0, && p = 1, 2, \ldots, P_k,\ k = 1, 2, \ldots, K \\
&\quad y_\ell \ge 0, && \ell = 1, 2, \ldots, L \\
&\quad r \ge 0.
\end{aligned}
$$

33

Minimum cost routing: non-splittable flow

- Declare each path (u_{kp}) as 0/1 option; thus, only one path to be chosen for each pair

$$\sum_{p=1}^{P_k} u_{kp} = 1, \quad k = 1, 2, \ldots, K.$$

- Optimization formulation: integer multi-commodity flow problem (4.5.3)

$$
\begin{aligned}
&\textbf{\textit{minimize}}_{\{u\}} \quad F = \sum_{k=1}^{K} \sum_{p=1}^{P_k} \xi_{kp} h_k u_{kp} \\
&\textbf{\textit{subject to}} \\
&\quad \sum_{p=1}^{P_k} u_{kp} = 1, && k = 1, 2, \ldots, K \\
&\quad \sum_{k=1}^{K} \sum_{p=1}^{P_k} \delta_{kp\ell}\, h_k u_{kp} \le c_\ell, && \ell = 1, 2, \ldots, L \\
&\quad u_{kp} = 0 \text{ or } 1, && p = 1, 2, \ldots, P_k,\ k = 1, 2, \ldots, K.
\end{aligned}
$$

34

- So far, we have done link-path based formulation
- Node-arc based formulation is possible: to show for the load balancing problem
  - Need to consider the flow on a link for a particular demand
  - Need to consider link-path indicator in terms of node incidence

35

TABLE 4.4 Notation used in the node-link formulation.

Notation	Explanation
Given:	
N	Number of nodes (indexed by $v = 1, 2, \ldots, N$)
K	Number of demand pairs with positive demand volume
L	Number of links
h_k	Demand volume of demand identifier $k = 1, 2, \ldots, K$
s_k	Source node of demand identifier $k = 1, 2, \ldots, K$
t_k	Destination node of demand identifier $k = 1, 2, \ldots, K$
c_ℓ	Capacity of link $\ell = 1, 2, \ldots, L$
$a_{v\ell}$	~~Link-path indicator, set to 1 if path p for demand pair k uses the link ℓ; 0, otherwise~~
$b_{v\ell}$	~~Link-path indicator, set to 1 if path p for demand pair k uses the link ℓ; 0, otherwise~~
Variables:	
$z_{\ell k}$	Flow amount on link ℓ for demand k
y_ℓ	Link-flow variable for link ℓ
r	Maximum link utilization variable

- a_{vl} := 1 if link l originates at node v; 0, otherwise
- b_{vl} := 1 if link l terminates at node v; 0, otherwise

36

Load balancing in node-link formulation (4.4.12)

$$
\begin{aligned}
&\boldsymbol{minimize}_{\{z,y,r\}} \quad F = r \\
&\boldsymbol{subject\ to} \\
&\qquad \sum_{\ell=1}^{L} a_{v\ell} z_{\ell k} - \sum_{\ell=1}^{L} b_{v\ell} z_{\ell k} = \begin{cases} h_k, & \text{if } v = s_k \\ 0, & \text{if } v \neq s_k, t_k, \quad v = 1, 2, \ldots, V \\ -h_k, & \text{if } v = t_k, \end{cases} \\
&\qquad\qquad\qquad\qquad\qquad\qquad k = 1, 2, \ldots, K \\
&\qquad \sum_{k=1}^{K} z_{\ell k} = y_\ell, \qquad \ell = 1, 2, \ldots, K \\
&\qquad y_\ell \le c_\ell r, \qquad \ell = 1, 2, \ldots, K \\
&\qquad z_{\ell k} \ge 0, \qquad \ell = 1, 2, \ldots, L, \ k = 1, 2, \ldots, K \\
&\qquad y_\ell \ge 0, \qquad \ell = 1, 2, \ldots, L \\
&\qquad r \ge 0.
\end{aligned}
$$

37

Summary

- Multi-commodity network flow
  - Started with a single-commodity example for a 3-node network
    - Three different objectives considered
      - Linearization of non-linear delay objective discussed
    - Solutions compared
  - Multi-commodity example for a 3-node network
  - The general case: link-path formulation and node-link formulation
- Linear programming solver: CPLEX
  - How to handle integer linear programming problems

38