Lecture 04

Date: 10-02-2020
Hydraulically favorable cross section

- Given the bottom slope $S_0$, the cross-sectional area $A$, and the roughness coefficient $n$.
- The cross section where the flow rate ($Q$) can flow the most. It is called an advantageous cross section.

This condition

$$Q = \frac{1}{n} AR^{2/3} S_0^{1/2}$$

$$\frac{\partial Q}{\partial h} = \frac{1}{n} AR^{-1/3} S_0^{1/2} \frac{\partial R}{\partial h} = 0$$

That is,

$$\frac{\partial R}{\partial h} = 0$$

from the maximum.

Or

$$\frac{\partial R}{\partial h} = \frac{\partial}{\partial h} \left( \frac{A}{P} \right) = -\frac{A}{P^2} \frac{\partial P}{\partial h} = 0$$

That is, $\frac{\partial P}{\partial h} = 0$

P is small = Few resistance.

Is obtained from the minimum.
In a rectangular section,

\[ P = b + 2h \]

\[ P = \frac{A}{h} + 2h \]

\[ \frac{\partial P}{\partial h} = -\frac{A}{h^2} + 2 = -\frac{bh}{h^2} + 2 = 0 \]

\[ \therefore b = 2h \]

In this case, an advantageous cross section is obtained.
In trapezoidal cross section,

\[ P = b + \frac{2h}{\sin \theta} = \frac{1}{h} \left( A - \frac{h^2}{\tan \theta} \right) + \frac{2h}{\sin \theta} \]

\[
\frac{\partial P}{\partial h} = -\frac{A}{h^2} - \frac{1}{\tan^2 \theta} + \frac{2}{\sin \theta}
\]

\[
= -\frac{1}{h^2} \left( bh + \frac{h^2}{\tan \theta} \right) - \frac{1}{\tan^2 \theta} + \frac{2}{\sin \theta}
\]

\[
= -\frac{b}{h} - \frac{2}{\tan \theta} + \frac{2}{\sin \theta} = 0
\]

\[
-\frac{b}{h} - \frac{2}{\tan \theta} + \frac{2}{\sin \theta} = 0
\]

\[
\Rightarrow \frac{b}{h} = -\frac{2}{\tan \theta} + \frac{2}{\sin \theta}
\]

\[
= 2 \frac{1 - \cos \theta}{\sin \theta} = 2 \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = 2 \tan \frac{\theta}{2}
\]

Therefore

\[
\therefore b = 2h \tan \frac{\theta}{2}
\]

Is advantageous in section.
$$F_f = \tau_0 PL$$
When the angle of bottom slope $\theta$ is small, $\sin \theta \approx \tan \theta$. Also, $\tan \theta = S_0$ and when the flow is uniform, $S_0 = S_f$.

Therefore, the active component of the gravity force is $W \sin \theta = \gamma AL \sin \theta \approx \gamma AL \tan \theta = \gamma ALS_0$, where $\gamma$ is the specific weight of water, $A$ is the cross-sectional area and $S_0$ is the channel bottom slope.

$$\gamma ALS_0 = \tau_0 PL$$

$$\tau_0 = \gamma \frac{A}{P} S_0 = \gamma RS_0 = \rho g RS_0$$
When the flow is uniform, \( S_f = S_0 \), \( \tau_0 = \gamma RS_0 = \rho g RS_0 \).

For a wide channel, \( R \approx h \) \( \tau_0 = \gamma h S_0 = \rho gh S_0 \).

The quantity \( \sqrt{\tau_0 / \rho} \) has the dimensions of velocity and the bed shear stress \( \tau_0 \) is expressed as:

\[
\tau_0 = \rho u^*^2 \quad \text{Or} \quad u^* = \sqrt{\frac{\tau_0}{\rho}}
\]

where \( u^* \) is known as the shear or friction or drag velocity, since it varies with the boundary friction \( \tau_0 \).

- It does not represent a velocity which is physically real.
- However, it is used as the velocity scale in the study of velocity distribution in open channels.

\[
u_\ast = \sqrt{g RS_0}
\]

For a wide channel, \( R \approx h \)

\[
u_\ast = \sqrt{gh S_0}
\]
Boundary Layer

- Velocity distribution across a channel section is not uniform owing to boundary roughness.
- Boundary layer is flow layer above the boundary, the flow velocities are retarded.
- Generally, thickness of the boundary layer is defined as the distance from the boundary surface to the point where \( u = 0.99u_0 \), when \( u_0 \) is the velocity in the outer layer.
- Almost all open channel flows are turbulent and the turbulent boundary layer intersects the free surface.
- Thus, the depth of flow \( h \) is the thickness of the boundary layer. The distance required to obtain a boundary layer thickness equal to the depth of turbulent open channel flow (i.e. \( \delta = h \)) is given by

\[
\frac{x}{h} = 3 \left[ \frac{\bar{U} h}{v} \right]^{0.25}
\]

where \( \bar{U} \) is the depth-averaged velocity.
Boundary Layer

Laminar or Viscous Sublayer

- Even in a turbulent boundary layer, there is a very thin layer near the boundary in which the flow is laminar and is known as the laminar or viscous sublayer.

\[ \delta_v = \frac{11.6 \nu}{u^*} \]
Fig. 8-1. Development of the boundary layer in an open channel with an ideal entrance condition.
Fig. 8-2. Distribution of velocity over a smooth channel surface (not in scale).
Fig. 8-3. Nature of surface roughness. (a) Smooth; (b) wavy; (c) rough.
Fig. 8-4. Sketches showing concept of three basic types of rough-surface flow: (a) isolated-roughness flow; (b) wake-interference flow; (c) quasi-smooth flow.
As stated earlier, velocity distribution across a channel section is not uniform owing to the presence of boundary (wall) roughness.

The effect of boundary roughness on the velocity distribution in turbulent flow was first investigated for pipe flow by Nikuradse who introduced the concept of equivalent sand grain roughness ($k_s$) as standard for all other types of roughness elements.

The ratio $k_s/R$ of the roughness height to the hydraulic radius is known as the relative roughness.

The roughness element mainly influence the velocity distribution close to surface generating eddies which affect the turbulence structure close to the boundary.
Schlichting (1968), in connection with flat surfaces and pipes, experimentally determined the following criteria, based on the ratio of Nikuradse roughness \((k_s)\) and a length scale of the viscous sublayer \((v/u_*)\), for classifying boundary surfaces:

1. Hydraulically smooth boundary:  
\[
\frac{u_*k_s}{v} \leq 5
\]

2. Hydraulically rough boundary:  
\[
\frac{u_*k_s}{v} \geq 70
\]

3. Transition boundary:  
\[
5 < \frac{u_*k_s}{v} < 70
\]
Table 4.1 Approximate values of $k_s$ for various materials (Chow, 1959)

<table>
<thead>
<tr>
<th>Material</th>
<th>$K_s$ range, ft</th>
<th>$K_s$ range, m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brass, Copper, Lead, Glass</td>
<td>0.0001-0.0030</td>
<td>0.00003-0.0009</td>
</tr>
<tr>
<td>Wrought iron, steel</td>
<td>0.0002-0.0080</td>
<td>0.00006-0.002</td>
</tr>
<tr>
<td>Asphaltered cast iron</td>
<td>0.0004-0.0070</td>
<td>0.0001-0.002</td>
</tr>
<tr>
<td>Galvanized iron</td>
<td>0.0005-0.0150</td>
<td>0.0002-0.0046</td>
</tr>
<tr>
<td>Cast iron</td>
<td>0.0008-0.180</td>
<td>0.0002-0.0055</td>
</tr>
<tr>
<td>Wood stave</td>
<td>0.0006-0.0030</td>
<td>0.0002-0.0009</td>
</tr>
<tr>
<td>Cement</td>
<td>0.0013-0.0040</td>
<td>0.0004-0.001</td>
</tr>
<tr>
<td>Concrete</td>
<td>0.0015-0.0100</td>
<td>0.0005-0.003</td>
</tr>
<tr>
<td>Drain title</td>
<td>0.0020-0.0100</td>
<td>0.0006-0.003</td>
</tr>
<tr>
<td>Riveted steel</td>
<td>0.0030-0.0300</td>
<td>0.0009-0.009</td>
</tr>
<tr>
<td>Natural river bed</td>
<td>0.1000-3.000</td>
<td>0.03-0.9</td>
</tr>
</tbody>
</table>
Velocity at a point in turbulent flow
Fig. 2.7 Definition of turbulent fluctuation in flow velocity.

(Bridge, 2003)
Fig. 2.8 Definition of turbulent motions in a boundary layer.

(Bridge, 2003)
To determine the variation of shear stress over the depth, let us consider the forces on a fluid element of a steady uniform flow as shown in Fig. below.

The hydrostatic pressure forces $F_{p1}$ and $F_{p2}$ are equal, since the water depths and the cross-sectional areas at the faces 1 and 2 are identical.

The gravity component in the x-direction is

$$F_g = \rho g \Delta A (h - z) \sin \theta$$

Equilibrium of forces in the x-direction yields

$$\tau_z \Delta A = F_g$$

$$\tau_z \Delta A = \rho g \Delta A (h - z) \sin \theta$$

$$\Rightarrow \tau_z = \rho g (h - z) \sin \theta = \rho g (h - z) S_0$$

Above Equation expresses a linear shear stress distribution over the depth.

For $z = 0$, $\tau_z = \tau_0$

$$\tau_0 = \rho gh S_0$$
\[ \tau_v = \text{Viscous shear stress} \]
\[ \tau_t = \text{turbulent shear stress} \]
\[ \tau_0 = \tau_t + \tau_v \]
According to Reynolds procedure, the shear stress at any height \( z \) in steady uniform flow can be described as

\[
\tau_z = \tau_v + \tau_t = \rho v \frac{du}{dz} - \rho u' w'
\]

- Viscous shear stress

\[
\tau_v = \rho v \frac{du}{dz}
\]

- Turbulent shear stress

\[
\tau_t = -\rho u' w'
\]
Although the time-averaged values of the vertical velocity and the turbulent flow fluctuations in $x$ and $y$ directions are equal to zero, the vertical turbulent flow fluctuations are not equal to zero ($w' \neq 0$)

and the time-averaged value of the product of these turbulent flow fluctuations is not equal to zero $\overline{u'w'} \neq 0$

and hence the turbulent component of the shear stress $\tau_t \neq 0$.

$$\tau_z = \rho g (h - z) S_0$$

$$\tau_z = \tau_v + \tau_t = \rho v \frac{du}{dz} - \rho \overline{u'w'}$$

$$\tau_z = \rho v \frac{du}{dz} - \rho \overline{u'w'} = \rho g (h - z) S_0$$

Thus, in uniform flow the shear stress increases linearly with the flow depth, which is due to the fact that the gravity force component in the direction of flow ($F_g$) increases linearly with the flow depth.

The contribution of the viscous shear stress is only of importance near the bed.
Prandtl Mixing Length Theory

- Prandtl, in an effort to relate the transport of momentum to the mean flow characteristics, introduced a characteristic length, $l$, which is termed the mixing length. Prandtl claimed that

$$u' \approx w' \approx l \left| \frac{du}{dz} \right|$$

$$\tau_t = -\rho u' w' = -\rho l^2 \left( \frac{du}{dz} \right)^2$$

$$\tau_t = \rho \varepsilon \frac{du}{dz}$$

where, $\varepsilon = l^2 \frac{du}{dz}$

Boussinesq equation

- $\varepsilon$ is known as the eddy viscosity similar to the kinematic viscosity $\nu$ and is a direct measure of the transport or mixing capacity of a turbulent flow.
Further, Prandtl assumed that the mixing length $l$ is proportion to the distance from the surface, i.e.

$$l = k z$$

- where $k$ is a proportionality factor called the von Karman turbulence constant.
- The value of $k$ has been determined by many experiments to be about 0.4 although there is some evidence that it may vary over a range of values depending on the Reynolds number of the flow.
Velocity Distribution in Laminar or Viscous Sublayer

- The viscous shear stress is dominant in this layer and the turbulent fluctuations can be neglected.
- In addition, the shear stress in this layer can be considered constant and equal to the bottom shear stress.
- Hence
  \[ \tau_0 = \tau_v = \rho v \frac{d\bar{u}}{dz} = \rho \overline{u}^2 \]
  or
  \[ \frac{d\bar{u}}{dz} = \frac{\overline{u}^2}{v} \]
- Integration of the above equation with the boundary condition \( u = 0 \) when \( z = 0 \) yields
  \[ u_z = \frac{\overline{u}^2 z}{v} \]
  \[ \therefore u_z \propto z \]
- which indicates that the velocity distribution in the laminar Sublayer is linear.
- The linear velocity distribution and the logarithmic velocity distribution intersect at a distance of \( \overline{u}^* z / v \) equal to 11.6, yielding a theoretical laminar sub layer thickness given by
  \[ \delta_v = \frac{11.6 v}{\overline{u}^*} \]
- However, measurements suggest that the fully viscous layer is somewhat smaller, equal to \( 5v/\overline{u}^* \).
\[
\frac{du}{dz} = \frac{u^*}{v}
\]

\[
\int du = \int \frac{u^*}{v} dz
\]

\[\Rightarrow u = \frac{u^*}{v} z + C\]

at \(u = 0, z = 0\) \(\Rightarrow C = 0\)

\[\Rightarrow u = \frac{u^*}{v} z\]
Velocity Distribution in Turbulent Boundary Layer

- Viscous effects can be neglected in this layer.
- Based on measurements, it is assumed that the turbulent shear stress in this layer is constant and equal to the bottom shear stress $\tau_0$.

$$\tau_0 = \tau_t = -\rho u'w' = \rho u^2$$

$$l = k z$$

$$\tau_t = -\rho u'w' = -\rho l^2 \left( \frac{du}{dz} \right)^2$$

$$\Rightarrow \rho u^2 = -\rho (kz)^2 \left( \frac{du}{dz} \right)^2$$

$$\Rightarrow \frac{du}{dz} = \frac{u^*}{kz}$$

By integrating

$$\frac{u_z}{u^*} = \frac{1}{k} \ln \frac{z}{z_0} \quad \therefore u_z \propto \ln z$$

- where $z_0$ is the zero velocity level, i.e. $u = 0$ at $z = z_0$ and depends on whether the boundary is hydraulically smooth or rough.
- Above Eq. indicates that the velocity distribution in turbulent flow is logarithmic and is commonly known as the Prandtl von Karman universal velocity distribution law.
\[
\frac{du}{dz} = \frac{u^*}{kz} \\
\int du = \int \frac{u^*}{kz} \, dz \\
\Rightarrow u = \frac{u^*}{k} \ln z + C \\
\text{at } u = 0, z = z_0 \quad \text{then } C = -\frac{1}{k} \ln z_0 \\
\Rightarrow u = \frac{u^*}{k} \ln z - \frac{1}{k} \ln z_0 \\
\Rightarrow \frac{u}{u^*} = \frac{1}{k} \ln \frac{z}{z_0}
\]
\[ \frac{u_z}{u^*} = \frac{1}{k} \ln \frac{z}{z_0} \] 

Equation (4.40) can also be written as

\[ \frac{u_z}{u^*} = \frac{1}{k} \ln \frac{z}{k_s} + B \]

where \( k_s \) is the Nikuradse roughness parameter.

Based on experiment, Nikuradse found

\[ B = \frac{1}{k} \ln \frac{u^* k_s}{v} + 5.50 \]

For hydraulically smooth boundary

\[ B = 8.5 \]

For hydraulically rough boundary

\[ \frac{u_z}{u^*} = \frac{1}{k} \ln \frac{z}{k_s} + \frac{1}{k} \ln \frac{u^* k_s}{v} + 5.50 \]

\[ \frac{u_z}{u^*} = \frac{1}{k} \ln \frac{z}{k_s} + 8.5 \]
For hydraulically smooth flow

\[ \frac{u_z}{u_*} = \frac{1}{k} \ln \frac{zu_*}{v} + 5.50 = \frac{1}{k} \ln \frac{9zu_*}{v} = \frac{1}{k} \ln \frac{z}{0.11v/u_*} \] .............(4.44)

For hydraulically rough flow

\[ \frac{u_z}{u_*} = \frac{l}{k} \ln \frac{z}{k_s} + 8.5 = \frac{1}{k} \ln \frac{30z}{k_s} = \frac{1}{k} \ln \frac{z}{0.033k_s} \] .................(4.45)

The velocity distribution over a boundary in the transition region may then be written as

\[ \frac{u_z}{u_*} = \frac{1}{k} \ln \frac{z}{(0.11v/u_* + 0.033k_s)} \] .................(4.46)

Experimental evidence suggests that the logarithmic velocity profile is a good approximation for the full depth of the flow.
\[
\frac{u_z}{u^*} = \frac{1}{k} \ln \frac{z}{z_0} \quad \text{..........................(4.40)}
\]

\[
\frac{u_z}{u^*} = \frac{1}{k} \ln \frac{z u^*_k}{v} + 5.50 = \frac{1}{k} \ln \frac{9 z u^*_k}{v} = \frac{1}{k} \ln \frac{z}{0.11 v / u^*_k} \quad \text{............(4.44)}
\]

\[
\frac{u_z}{u^*} = \frac{1}{k} \ln \frac{z}{k_s} + 8.5 = \frac{1}{k} \ln \frac{30 z}{k_s} = \frac{1}{k} \ln \frac{z}{0.033 k_s} \quad \text{......................(4.45)}
\]

\[
\frac{u_z}{u^*} = \frac{1}{k} \ln \frac{z}{(0.11 v / u^*_k + 0.033 k_s)} \quad \text{..................(4.46)}
\]

Comparing Eq. (4.40) with Eqs. (4.44), (4.45) and (4.46), the following values of \( z_0 \) are obtained.

1. Hydraulically smooth surface \( (u^*k_s/v \leq 5) \)

\[
z_0 = 0.11 \frac{v}{u^*} \quad \text{..............................(4.47)}
\]

2. Hydraulically rough surface \( (u^*k_s/v \geq 70) \)

\[
z_0 = 0.033 k_s \quad \text{..............................(4.48)}
\]

3. Transition regime \( (5 < u^*k_s/v < 70) \)

\[
z_0 = 0.11 \frac{v}{u^*} + 0.033 k_s \quad \text{..............................(4.49)}
\]
As stated earlier, the generally accepted procedure to determine the depth averaged velocity in a stream using the area-velocity method is to average the velocity measurements at 0.2h and 0.8h depths from the free surface, or alternatively, to measure velocity at 0.6h depth from the free surface, when h is the total depth of flow.

For logarithmic velocity distribution, Eq. (4.40), Vanoni (1941) showed that the flow velocity, measured at 0.632h from the free surface, is equal to the depth-averaged velocity in the vertical.
Keulegan (1938) integrated Eqs. (4.44) and (4.45) to get the mean velocity for turbulent flow in open channels and derived the following equations:

- **Hydraulically smooth surface** \((u^*k_s/v \leq 5)\)
  \[
  \frac{U}{u^*} = 5.75 \log \frac{u^* R}{v} + 3.25 = 5.75 \log \left( \frac{3.64 u^* R}{v} \right) \quad \text{.........(4.54)}
  \]

- **Hydraulically rough surface** \((u^*k_s/v \geq 70)\)
  \[
  \frac{U}{u^*} = 5.75 \log \frac{R}{k_s} + 6.25 = 5.75 \log \left( \frac{12.2 R}{k_s} \right) \quad \text{.........(4.55)}
  \]

- **Transition regime** \((5 < u^*k_s/v < 70)\)
  \[
  \frac{U}{u^*} = 5.75 \log \left( \frac{12.2 R}{k_s + 3.35 v/u^*} \right) \quad \text{.........(4.56)}
  \]

Equation (4.56) is known as the White-Colebrook formula which is basically valid for \(7.5 < R/k_s < 250\) (experimental range of Nikuradse).

Later study by lwagaki (1954) using data from various sources indicated that the values of 3.25 and 6.25 in Eqs. (4.54) and (4.55) are not truly constant for open channel flows, but vary with the Froude number since resistance to turbulent flow in open channels becomes larger than that in pipes due to increased instability of the free surface at high Froude numbers.
Example 4.1: A rectangular channel is 6 m wide and laid on a slope of 0.25%. The channel is made of concrete \((k_s = 0.002 m)\) and carries water at a depth of 0.50m. Compute the mean velocity of flow.

- \(S_0 = 0.25/100 = 0.0025\)
- \(R = A/P = (6 \times 0.50)/(6 + 2 \times 0.50) = 0.4286 m\)
- \(u^* = \sqrt{gRS_0} = \sqrt{9.81 \times 0.4286 \times 0.0025} = 0.1025 m/s\)
- \(k_s u^* = 0.002 \times 0.1025 \times 10^{-6} = 205 > 70\)
- Hence, the boundary is hydraulically rough and the mean velocity of flow is obtained by Eq. (4.55), i.e.

\[\frac{U}{u^*} = 5.75 \log \frac{12.2R}{k_s} = 5.75 \log \frac{12.2 \times 0.4286}{0.002} = 19.65\]

\[\therefore U = 19.65 \times 0.1025 = 2.014 m/s\]