Investigation of order of singularity in 3D two-phase transversely isotropic piezoelectric dissimilar joints by eigen analysis method


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Abstract
The distribution of stress singularity field near the vertex of two-phase material components is very important to maintain the reliability of intelligent materials. Piezoelectric material, due to its characteristic direct-converse piezoelectric effect, has naturally received considerable attentions. In this paper, stress singularity at a vertex in transversely isotropic piezoelectric dissimilar material joints is analyzed. Eigen analysis based on FEM is used for stress singularity field analysis of piezoelectric dissimilar material joints. The Eigen equation is used for calculating the order of stress singularity, and the angular function of elastic displacement, electric potential, stress and electric displacement. Stress and electric displacement fields demonstrate that the value near the free edge of the joint is larger than that at the inner portion. From the numerical result, it is suggested that delamination of the interface may occur at the interface edge of the dissimilar piezoelectric material joints due to the higher stress concentration at the free edge.

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1. Introduction
Piezoelectric material, due to its characteristic direct-converse piezoelectric effect, has naturally received considerable attentions [1]. Piezoelectric materials playing a key role as active components in many fields of
engineering and technology such as electronics, laser, microwave infrared, navigation and biology [2]. Mechanical stress occurs in piezoelectric material for any electric input. The stress concentrations caused by mechanical or electric loads may lead to crack initiation and extension, and sometimes the stress concentrations may be high enough to fracture the material parts.

Wang has obtained the general solutions of governing equations to three-dimensional axisymmetric problems in transversely isotropic piezoelectric media [3]. Williams used the mathematical procedure for analyzing stress singularities in infinite wedges and successfully applying to the analysis of stress distribution at the vicinity of a crack tip [4,5]. Aksentian determined Eigen values and Eigen vectors at the singular point in plane intersecting a free edge of the interface in three dimensional dissimilar joints [6]. Bazent and Estenssoro developed a finite element analysis for solving Eigen value equation to determine directly the order of stress singularity and the angular variation of the stress and displacement fields. This Eigen analysis was used to evaluate the order of singularity at a point where a crack meets a free surface in an isotropic material [7]. Then, this Eigen analysis based on a finite element was adapted by Pageau, Joseph and Biggersto use for analyzing the in-plane deformation of wedges and junctions composed of anisotropic materials. The stress and displacement fields were obtained from Eigen formulation for real and complex orders of stress singularity [8].

Stress singularity is related to debond and delamination at the interface edge of the bonded intelligent materials. The order of stress singularity near the vertex in 3D two-phase transversely isotropic piezoelectric dissimilar joints has not been made clear until now. Therefore, in this paper, the characteristic of singular stress distribution at the interface of transversely isotropic piezoelectric dissimilar Joints is investigated.

2. Formula of Analysis

In the absence of body forces and free charges, the equilibrium equations are expressed as follows [9]:

\[
\sigma_{ij,i} = 0, \quad d_{i,i} = 0
\]

(1)

where \(\sigma_{ij}\) represents the stress and \(d_i\) the electric displacement. The constitutive relations are shown as follows:

\[
\sigma_{ij} = c_{ijkl} \varepsilon_{kl}, \quad d_i = e_{ijkl} \varepsilon_{kl} - \chi_{ik} E_k
\]

(2)

where \(c_{ijkl}\) is the elastic constant, \(e_{ijkl}\) and \(\chi_{ik}\) are the piezoelectric constant and electric permittivity (dielectric constant), respectively. The elastic strain-displacement and electric field-potential relations are presented as follows:

\[
\varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \quad E_i = -\psi_i
\]

(3)

where \(\varepsilon_{ij}\) is the strain, and \(E_i\) is the electric field. \(u_i\) is the elastic displacement and \(\psi_i\) is the electric potential.

For transversely isotropic material, taking z-axis parallel to the poling axis of the material, by convention, the constructive relation is expressed in the following form.

\[
\mathbf{d}^T \begin{bmatrix} d^T_{ij} \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \varepsilon \end{bmatrix} = \begin{bmatrix} E \end{bmatrix}
\]

(4)

where \(\mathbf{d}^T_{ij}\) and \(\mathbf{d}^T\) are the stress and strain which are the mechanical field variables, \(\mathbf{d}^T\) and \(\mathbf{d}^T_e\) are the electric displacement and electric field respectively. Figure 1 represents the geometry of a typical case where a singular stress state occurs at the point o. The region surrounding the singular point is divided into a number of quadratic pyramidal elements with a summit o, with each element being located in spherical coordinates \(r, \theta, \phi\) by its nodes 1 to 8. A point P in the element can be located using the singular transformation by the relations.
\[ r = r_o \left( \frac{1 + \alpha}{2} \right)^{\rho} \quad \text{or,} \quad \rho = r \frac{r_o}{(1 + \alpha/2)^{\rho}} \quad (5) \]

\[ \theta = \sum_{i=1}^{8} H_i \theta_i \quad \text{and} \quad \phi = \sum_{i=1}^{8} H_i \phi_i \quad (6) \]

Where,

\[ H_1 = -\frac{1}{4} (1 - \eta)(1 - \xi)(\eta + \xi + 1) \quad H_2 = \frac{1}{2} (1 - \eta)(1 - \xi) \]

\[ H_3 = -\frac{1}{4} (1 - \eta)(1 + \xi)(\eta + \xi - 1) \quad H_4 = \frac{1}{2} (1 - \eta)(1 + \xi) \]

\[ H_5 = \frac{1}{4} (1 + \eta)(1 + \xi)(\eta + \xi - 1) \quad H_6 = \frac{1}{2} (1 + \eta)(1 - \xi) \]

\[ H_7 = \frac{1}{4} (1 + \eta)(1 - \xi)(\eta - \xi - 1) \quad H_8 = \frac{1}{2} (1 + \eta)(1 + \xi) \]

\( \theta \) and \( \phi \) are the nodal values of the angular co-ordinates and \( \alpha, \eta, \) and \( \xi \) are natural co-ordinates of the element whose ranges are shown in Fig.1. \( H \) interpolation function, \( p \) Eigen value, \( \rho = r/r_o, r \) the distance from the singular point.

\[ \text{Fig. 1. Element geometry and natural co-ordinates at a free edge singular point} \]

The elastic displacement and electric potential field in the element is expressed as follows:

\[ \begin{align*} 
\bar{u} - \bar{u}_o &= \left( \frac{1 + \alpha}{2} \right) \sum_{i=1}^{8} H_i \left( \bar{u}_i - \bar{u}_o \right), \quad \text{and} \quad \bar{\psi} - \bar{\psi}_o &= \left( \frac{1 + \alpha}{2} \right) \sum_{i=1}^{8} H_i \left( \bar{\psi}_i - \bar{\psi}_o \right) 
\end{align*} \]

(7)

where \( \bar{u}_o \) and \( \bar{u} \) represents the elastic displacement vector of the vertex \( o \) and the point \( P \) respectively, and \( \bar{u}_i \) represents the elastic displacement vector of the node \( i \) \( (i = 1, 2, \ldots, 8) \). Similarly \( \bar{\psi}_o \) and \( \bar{\psi} \) represents the electric potential vector of the vertex \( o \) and the point \( P \) respectively, and \( \bar{\psi}_i \) represents the electric potential vector of the node \( i \) \( (i = 1, 2, \ldots, 8) \). In order to simplify the notation, the following equation can be defined.

\[ \begin{align*} 
\bar{u} &= \left( \bar{u} - \bar{u}_o \right), \quad \bar{u}_i = \left( \bar{u}_i - \bar{u}_o \right), \quad \text{and} \quad \bar{\psi} &= \left( \bar{\psi} - \bar{\psi}_o \right), \quad \bar{\psi}_i = \left( \bar{\psi}_i - \bar{\psi}_o \right) 
\end{align*} \]

(8)

Using the Eq. (5), Eq. (7) can be expressed as follows:
3. The Eigen Equation

The Eigen equation was formulated for determining the order of stress singularity as follows [8]:

\[
\begin{pmatrix}
\rho^2 [A] + \rho [B] + [C]
\end{pmatrix}\{U\} = \{0\}
\]

(10)

where, \(\{U\} = \begin{pmatrix} u_r \\ u_\theta \\ u_\phi \\ \psi \end{pmatrix}\), \([A] = \sum_s ((k_s - k_{ss}))\cdot [B] = \sum_s ((l_s - l_{ss}))\cdot [C] = \sum_s ((k_s - k_{ss}))\)

In Eq. (10), \(p\) represents the characteristic root, which is related to the order of singularity, \(\lambda\), as \(\lambda = 1 - p\). \([A]\), \([B]\) and \([C]\) are matrices composed of material properties, and \(\{U\}\) represents the elastic displacement and electric potential vector.

4. The Angular Function

The elastic displacement and electric potential equation is expressed by the following equation:

\[
u_j (r, \theta, \phi) = b_j (\theta, \phi)r^{\lambda-\frac{1}{2}}, \quad \psi_i (r, \theta, \phi) = q_i (\theta, \phi)r^{\lambda-\frac{1}{2}}
\]

(11)

By differentiating the above equations, get the angular function of strain and electric field equation respectively. The stress and electric displacement distribution equations in the stress singularity region can be expressed as:

\[
\sigma_j (r, \theta, \phi) = K_i r^{\lambda-\frac{1}{2}} f_j (\theta, \phi), \quad d_i \{\theta, \phi\}^{-\frac{1}{2}} F_i r^{\lambda-\frac{1}{2}} l_i \{\theta, \phi\}^{-\frac{1}{2}}
\]

(12)

Where \(r\) represents the distance from the stress singular point, \(b_j (\theta, \phi)\) the angular function of elastic displacement, \(q_i (\theta, \phi)\) the angular function of electric potential, \(f_j (\theta, \phi)\) the angular function of stress distribution, \(l_i (\theta, \phi)\) the angular function of electric displacement, \(K_i\) the intensity of singularity, \(F_i\) the intensity of electric field, and \(\lambda\) the order of stress singularity. Angular functions of stress components obtained from eigen analysis in Eq. (10) are examined.

5. Problem Description and Numerical Example

Figure 2(a) represents a model for 3D two-phase transversely isotropic piezoelectric dissipimilar joints used in the present analysis. The stress singularity line and singularity point on interface of the joint are shown in the figure. The angle \(\theta\) is equal to 180º and the angle \(\phi\) is equal to 90º. In Eigen analysis, a mesh division for the joint is needed for the analysis. The mesh developed on \(\phi, \theta\) plane is shown in Fig. 2(b), where the surface of a unit sphere is divided into \(\phi \times \theta = 10º \times 10º\) and the surface of a unit sphere near edge and interface is divided into \(\phi \times \theta = 1º \times 1º\).
Fig. 2. (a) Singular point of 3D piezoelectric joint in x, y, z plane; (b) a mesh on developed ϕθ plane

Table 1. Material properties of piezoelectric materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Elastic Constant, $10^{10}$ N/m$^2$</th>
<th>Piezoelectric Constant, C/m$^2$</th>
<th>Dielectric Constant, $10^{-10}$ C/Vm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c_{11}$</td>
<td>$c_{12}$</td>
<td>$c_{13}$</td>
</tr>
<tr>
<td>PZT-6B</td>
<td>16.8</td>
<td>6.00</td>
<td></td>
</tr>
<tr>
<td>PZT-7</td>
<td>13.0</td>
<td>8.30</td>
<td></td>
</tr>
</tbody>
</table>

The material properties of piezoelectric materials are shown in above table. In this analysis, first of all, Eigen values and Eigen vectors are investigated by Eigen analysis when two different materials are bonded. The order of singularity $\lambda$, at the vertex and at a point on the singularity line for the model shown in Fig. 2 is calculated. Solving Eigen equation yields many roots $p$ and Eigen vectors corresponding to each Eigen value are obtained. However, if the root $p$ is within the range of $0 < p < 1$, this fact indicates that the stress field has singularity. The values of the order of singularity at the singularity corner and line for piezoelectric similar material are shown in Table 2.

Table 2. Order of singularity for PZT-6B and PZT-7

<table>
<thead>
<tr>
<th></th>
<th>$p_{\text{vertex}}$</th>
<th>$p_{\text{line}}$</th>
<th>$\lambda_{\text{vertex}}$</th>
<th>$\lambda_{\text{line}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PZT-6B</td>
<td>0.0177</td>
<td>2.50</td>
<td>-10.3</td>
<td>3.5</td>
</tr>
<tr>
<td>PZT-7</td>
<td>0.9823</td>
<td>0.9822</td>
<td>171.0</td>
<td>186.0</td>
</tr>
</tbody>
</table>

Fig. 3. Distribution of $b_r$, $b_\theta$, $b_\phi$, $q$ against ϕ and θ for PZT-6B and PZT-7

The distributions of elastic displacement and electric potential are shown in fig. 3. It is found that, the angular function of elastic displacement and electric potential is continuous at the interface of the joints and has larger value at the free edge than the inner portion. It is also found that the angular function of elastic displacement and electric
potential at $\phi = 0^\circ$ is agreed with that at $\phi = 90^\circ$. The figure of $b_\phi$ and $b_r$ is symmetry with respect to $\phi$ and the figure of $b_\theta$ and $q$ is antisymmetry with respect to $\phi$.

The distributions of normalized angular function of strain and electric field with respect to the angle $\phi$ at $\theta = 90^\circ$ in the singular field for the piezoelectric bonded structure are shown in fig. 4. It is shown from the figure that the strain and electric field have larger value near the free edge of the bonded joints. The distributions of normalized angular function of stress and electric displacement with respect to the angle $\phi$ at $\theta = 90^\circ$ for the piezoelectric bonded structure are shown in fig. 5. These distributions are nearly similar to the distribution of angular function strain and electric field. It is shown from the figure that the stress and electric displacement have larger value near the free edge of the bonded joints. So there is another possibility to debond and delamination occurs near the free edge of the bonded joints.

6. Conclusion

In this paper, an Eigen analysis based on a finite element method formulation at the vertex of transversely isotropic piezoelectric dissimilar joint was presented. Angular functions of stress and electric displacement for singularities corner were derived from Eigen analysis. From the numerical investigation, the following conclusions can be drawn for the piezoelectric dissimilar bonded joints.

a) The higher angular function occurs near the interface edge than the inner portion of the bonded joint.

b) The possibility to debond and delamination at the interface edge of the piezoelectric dissimilar material joints was due to the higher stress and electric displacement concentration at the free edge.

References